

Super-renormalizable Multidimensional Quantum Gravity

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In this paper we introduce a perturbatively super-renormalizable and unitary theory of quantum gravity in any dimension D . In four dimensions the theory is an extension of the Stelle higher derivative gravity that involves an infinite number of derivative terms characterized by two entire functions, a.k.a. “form factors”. In dimension D we preserve two entire functions and we implement a finite number of local operators required by the quantum consistency of the theory. The main reason to introduce the entire functions is to avoid ghosts (states of negative norm) like the one in the four-dimensional Stelle’s theory. The new theory is indeed ghost-free since the two entire functions have the property to generalize the Einstein-Hilbert action without introducing new poles in the propagator. By expanding the form factors to the lowest order in a mass scale we introduce, the local high derivative theory is recovered. Any truncation of the entire functions gives rise to the unitarity violation and it is only by keeping all the infinite series that we overcome similar issues. The theory is renormalizable at one loop and finite from two loops upward. More precisely, the theory turns out to be super-renormalizable because the covariant counter-terms have less derivatives than the classical action and the coefficients of the terms with more derivatives do not need any kind of infinity renormalization. In this paper we essentially study three classes of form factors, systematically showing the tree-level unitarity.

We prove that the gravitation potential is regular in $r = 0$ for all the choices of form factors compatible with renormalizability and unitarity. We also include Black hole spherical symmetric solutions omitting higher curvature corrections to the equation of motions. For two out of three form factors the solutions are regular and the classical singularity is replaced by a “de Sitter-like core” in $r = 0$.

For one particular choice of the form factors, we prove that the D -dimensional “Newtonian cosmology” is singularity-free and the Universe spontaneously follows a de Sitter evolution at the “Planck scale” for any matter content (either dust or radiation).

We conclude the article providing an extensive analysis of the spectral dimension for any D and for the three classes of theories. In the ultraviolet regime the spectral dimension takes on different values for the three cases: less than or equal to “1” for the first case, “0” for the second one and “2” for the third one. Once the class of theories compatible with renormalizability and unitarity is defined, the spectral dimension has the same short distance “critical value” or “accumulation point” for any value of the topological dimension D .

In 1916 Albert Einstein revolutionized the way physicists thought about gravity with the theory of “general relativity”. This theory works quite well at the classical level, but at the theoretical level one of the biggest problems is to find a theory that is able to reconcile general relativity and quantum mechanics. There are many reasons to believe that gravity has to be quantum, some of which are: the quantum nature of matter in the right-hand side of the Einstein equations, the singularities appearing in classical solutions of general relativity, and so on.

Let us recall here the main hypothesis underlying general relativity. The grounding principles of “general relativity” are: (i) gravity is no longer a force as in the Newton’s theory, but it is the “curvature of the space-time”, (ii) the symmetry principle underlying the gravitational theory is the “general covariance” or “invariance under general coordinate transformations”, (iii) the “energy momentum tensor” for any kind of matter has to be covariantly conserved, (iv) the dynamics should be described by “second order” differential equations. Assuming these fundamental requirements, we get a “unique” theory for the dynamics of gravity, namely the Einstein equations. The theory can be also formulated starting

from an action principle by Einstein and Hilbert. The action principle we are going to introduce in this paper makes gravity compatible with quantum mechanics, as it is the result of a synthesis of minimal requirements beyond the classical Einstein-Hilbert action.

Let us axiomatically itemize these requirements one by one:

- (i). gravity is still retained as curvature of the space-time and the underlying symmetry principle remains “general covariance”;
- (ii). Einstein-Hilbert action is no longer the correct one, but it should still be a good approximation of the theory at an energy scale much smaller than the Planck mass or any other invariant scale;
- (iii). solutions of the classical equations of motion must be singularity-free;
- (iv). the theory has to be perturbatively renormalizable or finite at quantum level. In other words, we assume “axiomatically” the validity of the “perturbative theory”. We claim that any mathematical theory which prides itself to describe the universe must

be perturbatively consistent. This is empirically true for all the other fundamental forces: Weak, Strong and Electromagnetic. If for a general system we find that this is not the case, then we have to change the theory or “dualize” it [1–4] instead of trying to solve it at a non-perturbative level;

- (v). the spacetime spectral-dimension should decrease with the energy to obtain a complete quantum gravity theory in the ultraviolet regime (this hypothesis is strongly related to the previous one (iv)). This property seems to be of “universal nature” because it is common to many approaches to quantum gravity [5–16]. The Stelle’s theory [17], the Crane-Smolín theory [18], “asymptotically safe quantum gravity” [19–25], “causal dynamical triangulations” [15, 16], “loop quantum gravity” [8] and “string theory” already manifest this property with an high energy spectral dimension $d_s = 2$ or less;
- (vi). the theory has to be unitary, with no other pole except the graviton one in the propagator;
- (vii). spacetime is a single continuum of space and time and in particular the Lorentz invariance is not broken. This requirement is supported by recent observations.

The main hypothesis we abandon with respect to the classical theory is the absence of higher derivative terms in the action. As we are going to show we admit indeed an infinite tower of operators defined through an “entire function” of the D’Alembertian differential operator.

This work is inspired by papers in four dimensions about a “nonlocal extension” of gauge theories and gravity published in the Nineties [27, 32–39]. For example, in [33–39] the authors considered a modification of the Feynman rules, where the coupling constants (g_i for electro-weak interactions and G_N for gravity) become functions of the momentum p . They proved the gauge invariance at all orders in gauge theory but only up to the second order in gravity. For particular choices of $g_i(p)$ or $G_N(p)$, the propagators do not show any other pole besides those of the standard particle content of the theory, which makes the theory unitary. On the other hand the theory is also finite if the coupling constants go sufficiently fast to zero in the ultra-violet limit. On the basis of these conclusions, the problem with gravity remains to find a covariant action that self-contains the properties mentioned above: finiteness and/or renormalizability and unitarity.

We now introduce our theory starting from the perturbative D -dimensional “non-renormalizable” Einstein gravity, and recalling the four dimensional Stelle’s [17] quantum gravity, which serves as an example of power-counting renormalizable (but not unitary) theory.

First of all, let us explain why it is important to study quantum gravity in D -dimensions. There are at least five reasons to look for a super-renormalizable theory of gravity with extra dimensions. (i) The first reason is

that $D > 4$ eliminates “soft-graviton or infra-red divergences” at quantum level [40]; (ii) the second one is that only in $D \geq 7$ there exists a well defined “total cross section” [40]; (iii) the third one is to obtain a well defined “Kaluza-Klein grand-unification”; (iv) the fourth one is the possibility to have a well defined completion of the 11-dimensional supergravity as a candidate for “M-Theory”; (v) the last reason to study gravity in any spacetime dimension is related to the universality of the quantization procedure independently from the number of dimensions “ D ”. In synthesis, we do not want to tune the spacetime dimension to a particular value to make the theory consistent at quantum level; instead, the theory “should be” well defined for any value of D .

In this paper we use the signature $(+ - \dots -)$. The curvature tensor is defined by $R^\alpha_{\beta\gamma\delta} = -\partial_\delta \Gamma^\alpha_{\beta\gamma} + \dots$, the Ricci tensor by $R_{\mu\nu} = R^\alpha_{\mu\nu\alpha}$, and the curvature scalar by $R = g^{\mu\nu} R_{\mu\nu}$ and $g_{\mu\nu}$, where the metric tensor.

Perturbative quantum gravity is the quantum theory of a spin-two particle on a fixed (conventionally flat) background. Starting from the D -dimensional Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\text{EH}} = \sqrt{|g|} 2\kappa^{-2} R, \quad (1)$$

($\kappa^2 = 32\pi G_N$) we split the metric in a background part and in a fluctuation

$$g_{\mu\nu} = g^o_{\mu\nu} + \kappa h_{\mu\nu}. \quad (2)$$

Using (2), we then expand the action in powers of the graviton fluctuation $h_{\mu\nu}$ around the fixed background $g^o_{\mu\nu}$ that we assume to be the flat Minkowski metric $\eta_{\mu\nu}$. Regrettably, the quantum theory is divergent at two loops in $D = 4$, so we should introduce a counter-term proportional to the Riemann tensor at the third power

$$\mathcal{L}_{R^3} \propto G_N \sqrt{-g} R^{\alpha\beta}_{\gamma\delta} R^{\gamma\delta}_{\rho\sigma} R^{\rho\sigma}_{\alpha\beta}. \quad (3)$$

In the general D dimensional Einstein-Hilbert theory the superficial degree of divergence of a Feynman diagram is $\delta = L D + 2V - 2I$, where L is the number of loops, V is the number of vertices and I the number of internal lines in the graph. Substituting in δ the topological relation $L = 1 + I - V$, we obtain

$$\delta = 2 + (D - 2)L \quad \text{for } \mathcal{L}_{\text{EH}}. \quad (4)$$

In $D > 2$ the superficial degree of divergence increases with the number of loops. Therefore, we are forced to introduce an infinite tower of higher derivative counter-terms and an infinite number of coupling constants, thus making the theory not predictive.

Schematically, we can relate the number L of loop divergences to the counter terms we have to introduce to regularize the theory. In short

$$S = \int d^D x \sqrt{|g|} \left[2\kappa^{-2} R + \underbrace{\sum_{m,n}^{\infty} \frac{\alpha_{nm}}{\epsilon} \nabla^n R^m}_{n+2m=2+(D-2)L} \right], \quad (5)$$

where “ n ” and “ m ” are integer numbers, α_{mn} are coupling constants and $1/\epsilon$ is the cutoff in dimensional regularization.

Now we want to comment further about the above statement on the meaning of renormalizability. Scholars usually claim that a theory loses its predictability when its action consists of an infinity number of operators. Consequently, they believe that the theory can be defined only through an infinite number of measures. Such statement is highly questionable because we should always add all the possible operators to a Lagrangian and/or an Hamiltonian describing a physical system. The most important step we need to take is to assess whether the physical measurable quantities are affected or not by the above-mentioned operators. In other words it is essential to establish if such operators are “relevant or irrelevant” to the physical observable. If we can assume that no physical quantity is susceptible to such operators, we can then empirically infer that the coupling constants equal “zero”, as long as the theory satisfies the unitarity requirement. Let us provide an example. Suppose to add other hermitian operators to the standard Hamiltonian of the hydrogen atom. We can “invent” an infinite number of such operators but only few of them will be relevant, like for example the relativistic corrections or the Lamb Shift, while all the other ones are irrelevant. In this case what is really significant is the precision level in the measure of the energy spectrum, compatibly with a zero value for the other couplings, as long as unitarity is satisfied.

A first revolution in quantum gravity in $4D$ was introduced by Stelle [17, 41] with his higher derivative theory

$$S = \int d^4x \sqrt{-g} [\alpha R_{\mu\nu} R^{\mu\nu} - \beta R^2 + 2\kappa^{-2} R]. \quad (6)$$

If we calculate the upper bound to the superficial degree of freedom for this theory in D -dimensions we find

$$\delta = DL - 4I + 4V = D - (D - 4)(V - I), \quad (7)$$

so that $\delta = 4$ in $D = 4$. The theory is then renormalizable, since all the divergences can be absorbed in the operators already present in the Lagrangian (6). Unfortunately however, the propagator contains a physical ghost (state of negative norm) that represents a violation of unitarity. Probability, as described by the scattering S -matrix, is no longer preserved. Similarly, the classical theory is destabilized, since the dynamics can drive the system to become arbitrarily excited, and the Hamiltonian constraint is unbounded from below. On this basis, we can generalize the Stelle theory to a D -dimensional renormalizable one. In short, the Lagrangian with at most X derivatives of the metric is

$$\mathcal{L}_{D-\text{Ren}} = a_1 R + a_2 R^2 + b_2 R_{\mu\nu}^2 + \dots + a_x R^{X/2} + b_x R_{\mu\nu}^{X/2} + c_X R_{\mu\nu\rho\sigma}^{X/2} + d_X R \square^{\frac{X}{2}-2} R \dots \quad (8)$$

In the second line, the dots on the left imply a finite number of extra terms with fewer derivatives of the metric

tensor, and the dots on the right indicate a finite number of operators with the same number of derivatives but higher powers of the curvature ($O(R^2 \square^{X/2-4} R)$).

In this theory, the power counting tells us that the maximal superficial degree of divergence of a Feynmann graph is

$$\delta = D - (D - X)(V - I). \quad (9)$$

For $X = D$ the theory is renormalizable since the maximal divergence is $\delta = D$ and all the infinities can be absorbed in the operators already present in the action (8).

The general action of “derivative order N ” can be found combining curvature tensors and covariant derivatives of the curvature tensor. In short the action reads as follows [42],

$$S = \sum_{n=0}^{N+2} \alpha_{2n} \Lambda^{D-2n} \int d^D x \sqrt{|g|} \mathcal{O}_{2n}(\partial_\rho g_{\mu\nu}) + S_{\text{NL}}, \quad (10)$$

where Λ is a mass scale in our fundamental theory, $\mathcal{O}_{2n}(\partial_\rho g_{\mu\nu})$ denotes the general covariant scalar term containing “ $2n$ ” derivatives of the metric $g_{\mu\nu}$, while S_{NL} is a nonlocal action term that we are going to set later [45–48]. The maximal number of derivatives in the local part of the action is $2N + 4$. We can classify the local terms in the following way,

$$\begin{aligned} N = 0 : & \quad S_0 = \lambda + c_0^{(0)} R + c_1^{(0)} R^2 + c_2^{(0)} R_{\mu\nu} R^{\mu\nu}, \\ N = 1 : & \quad S_1 = S_0 + c_1^{(1)} R_{\dots}^3 + c_2^{(1)} \nabla R_{\dots} \nabla R_{\dots}, \\ N = 2 : & \quad S_2 = S_0 + S_1 + c_1^{(2)} R_{\dots}^4 \\ & \quad + c_2^{(2)} R_{\dots} \nabla R_{\dots} \nabla R_{\dots} + c_3^{(3)} \nabla^2 R_{\dots} \nabla^2 R_{\dots}, \\ & \quad \dots \\ & \quad \dots \\ N = N : & \quad S_N = \sum_{i=0}^{N-1, N>0} S_i + c_1^{(N)} R_{\dots}^{N+2} + \\ & \quad + c_2^{(N)} R_{\dots}^{N-1} \nabla R_{\dots} \nabla R_{\dots} + \dots + c_M^{(N)} R_{\dots} \square^N R_{\dots}. \end{aligned} \quad (11)$$

In the local theory (8), renormalizability requires $X = D$, so that the relation between the spacetime dimension and the derivative order is $2N + 4 = D$. To avoid fractional powers of the D’Alembertian operator, we take $2N + 4 = D_{\text{odd}} + 1$ in odd dimensions and $2N + 4 = D_{\text{even}}$ in even dimensions.

In this paper, we are focused on the renormalizability and unitarity of the theory, so the main quantity to calculate is the graviton propagator. Although the action is complicated, we only need to consider the quadratic operators in the curvature to get the “two points function”. Given the general structure (10), for $N > 0$ and $n > 2$ contributions to the propagator come only from the following operators,

$$R_{\mu\nu} \square^{n-2} R^{\mu\nu}, \quad R \square^{n-2} R, \quad R_{\mu\nu\alpha\beta} \square^{n-2} R^{\mu\nu\alpha\beta}. \quad (12)$$

However, using the Bianchi and Ricci identities one can reduce the terms listed above from three to two (with total $2n$ derivatives) [44],

$$\begin{aligned} R_{\mu\nu\alpha\beta}\square^{n-2}R^{\mu\nu\alpha\beta} \\ = -\nabla_\lambda R_{\mu\nu\alpha\beta}\square^{n-3}\nabla^\lambda R^{\mu\nu\alpha\beta} + O(R^3) + \nabla_\mu\Omega^\mu \\ = 4R_{\mu\nu}\square^{n-2}R^{\mu\nu} - R\square^{n-2}R + O(R^3) + \nabla_\mu\Omega'^\mu, \end{aligned} \quad (13)$$

where $\nabla_\mu\Omega^\mu$ and $\nabla_\mu\Omega'^\mu$ are total divergence terms. Applying (13) to (12), for $n > 2$ we discard the third term and we keep the first two.

We now have to define the “non-local” action term in (10). As we are going to show, both super-renormalizability and unitarity require the following two non-local operators,

$$R_{\mu\nu}h_2(-\square_\Lambda)R^{\mu\nu}, \quad Rh_0(-\square_\Lambda)R. \quad (14)$$

The full action, focusing mainly on the non-local terms and the quadratic part in the curvature, reads

$$\begin{aligned} S = \int d^Dx \sqrt{|g|} \Big[2\kappa^{-2}R + \bar{\lambda} \\ + \sum_{n=0}^N \left(a_n R(-\square_\Lambda)^n R + b_n R_{\mu\nu}(-\square_\Lambda)^n R^{\mu\nu} \right. \\ + R h_0(-\square_\Lambda) R + R_{\mu\nu} h_2(-\square_\Lambda) R^{\mu\nu} \\ \left. + \underbrace{\dots\dots\dots O(R^3) \dots\dots\dots + R^{N+2}}_{\text{Finite number of terms}} \right]. \end{aligned} \quad (15)$$

The last line is a collection of local terms that are renormalized at quantum level. In the action, the couplings and the non-local functions have the following dimensions: $[a_n] = [b_n] = M^{D-4}$, $[\kappa^2] = M^{2-D}$, $[\bar{\lambda}] = M^D$, $[h_2] = [h_0] = M^{D-2}$.

At this point, we are ready to expand the Lagrangian at the second order in the graviton fluctuation. Splitting the spacetime metric in the flat Minkowski background and the fluctuation $h_{\mu\nu}$ as defined in (2), we get [43]

$$\begin{aligned} \mathcal{L}_{\text{lin}} = -\frac{1}{2}[h^{\mu\nu}\square h_{\mu\nu} + A_\nu^2 + (A_\nu - \phi_{,\nu})^2] \\ + \frac{1}{4}\left[\frac{\kappa^2}{2}\square h_{\mu\nu}\beta(\square)\square h^{\mu\nu} - \frac{\kappa^2}{2}A_{\mu,\nu}^\mu\beta(\square)A_{,\nu}^\nu \right. \\ - \frac{\kappa^2}{2}F^{\mu\nu}\beta(\square)F_{\mu\nu} + \frac{\kappa^2}{2}(A_{,\alpha}^\alpha - \square\phi)\beta(\square)(A_{,\beta}^\beta - \square\phi) \\ \left. + 2\kappa^2(A_{,\alpha}^\alpha - \square\phi)\alpha(\square)(A_{,\beta}^\beta - \square\phi)\right], \end{aligned} \quad (16)$$

where $A^\mu = h^{\mu\nu}_{,\nu}$, $\phi = h$ (the trace of $h_{\mu\nu}$), $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ and the functionals of the D'Alembertian operator $\beta(\square)$, $\alpha(\square)$ are defined by

$$\begin{aligned} \alpha(\square)/2 &:= \sum_{n=0}^N a_n(-\square_\Lambda)^n + h_0(-\square_\Lambda), \\ \beta(\square)/2 &:= \sum_{n=0}^N b_n(-\square_\Lambda)^n + h_2(-\square_\Lambda). \end{aligned} \quad (17)$$

The d'Alembertian operator in (16) and (17) must be conceived on the flat spacetime. The linearized Lagrangian (16) is invariant under infinitesimal coordinate transformations $x^\mu \rightarrow x^\mu + \kappa\xi^\mu(x)$, where $\xi^\mu(x)$ is an infinitesimal vector field of dimensions $[\xi(x)] = M^{(D-4)/2}$. Under this transformation, the graviton field turns into

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \xi(x)_{\mu,\nu} - \xi(x)_{\nu,\mu}. \quad (18)$$

The presence of the local gauge symmetry (18) calls for the addition of a gauge-fixing term to the linearized Lagrangian (16). Hence, we choose the following fairly general gauge-fixing operator

$$\begin{aligned} \mathcal{L}_{\text{GF}} = \lambda_1(A_\nu - \lambda\phi_{,\nu})\omega_1(-\square_\Lambda)(A^\nu - \lambda\phi^{,\nu}) \\ + \frac{\lambda_2\kappa^2}{8}(A_{\mu,\nu}^\mu - \lambda\square\phi)\beta(\square)\omega_2(-\square_\Lambda)(A_{,\nu}^\nu - \lambda\square\phi) \\ + \frac{\lambda_3\kappa^2}{8}F_{\mu\nu}\beta(\square)\omega_2(-\square_\Lambda)F^{\mu\nu}, \end{aligned} \quad (19)$$

where $\omega_i(-\square_\Lambda)$ are three weight functionals [17, 48]. In the harmonic gauge $\lambda = \lambda_2 = \lambda_3 = 0$ and $\lambda_1 = 1/\xi$. The linearized gauge-fixed Lagrangian reads

$$\mathcal{L}_{\text{lin}} + \mathcal{L}_{\text{GF}} = \frac{1}{2}h^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}h^{\rho\sigma}, \quad (20)$$

where the operator \mathcal{O} is made of two terms, one coming from the linearized Lagrangian (16) and the other from the gauge-fixing term (19). Inverting the operator \mathcal{O} [43] we find the two point function in the harmonic gauge ($\partial^\mu h_{\mu\nu} = 0$),

$$\begin{aligned} \mathcal{O}^{-1}(k) = \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2\omega_1(k^2/\Lambda^2)} + \frac{P^{(2)}}{k^2\left(1 + \frac{k^2\kappa^2\beta(k^2)}{4}\right)} \\ - \frac{P^{(0)}}{2k^2\left(\frac{D-2}{2} - \frac{D\beta(k^2)\kappa^2/4 + (D-1)\alpha(k^2)\kappa^2}{2}\right)}. \end{aligned} \quad (21)$$

The tensorial indexes for the operator \mathcal{O}^{-1} and the projectors $P^{(0)}$, $P^{(2)}$, $P^{(1)}$, $\bar{P}^{(0)}$ have been omitted and the functions $\alpha(k^2)$ and $\beta(k^2)$ are achieved by replacing $-\square \rightarrow k^2$ in the definitions (17). The projectors are defined by [43, 63]

$$\begin{aligned} P_{\mu\nu\rho\sigma}^{(2)}(k) &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}, \\ P_{\mu\nu\rho\sigma}^{(1)}(k) &= \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}), \\ P_{\mu\nu\rho\sigma}^{(0)}(k) &= \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}, \quad \bar{P}_{\mu\nu\rho\sigma}^{(0)}(k) = \omega_{\mu\nu}\omega_{\rho\sigma}, \\ \bar{P}_{\mu\nu\rho\sigma}^{(0)} &= \theta_{\mu\nu}\omega_{\rho\sigma} + \omega_{\mu\nu}\theta_{\rho\sigma}, \\ \theta_{\mu\nu} &= \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}. \end{aligned} \quad (22)$$

Looking at the last two gauge invariant terms in (21), we deem convenient to introduce the following defini-

tions,

$$\begin{aligned}\bar{h}_2(z) &= 1 + \frac{\kappa^2 \Lambda^2}{2} z \sum_{n=0}^N b_n z^n + \frac{\kappa^2 \Lambda^2}{2} z h_2(z), \\ \left(\frac{D-2}{2}\right) \bar{h}_0(z) &= \frac{D-2}{2} - \frac{\kappa^2 \Lambda^2 D}{4} z \left[\sum_{n=0}^N b_n z^n + h_2(z) \right] \\ &\quad - \kappa^2 \Lambda^2 (D-1) z \left[\sum_{n=0}^N a_n z^n + h_0(z) \right],\end{aligned}\quad (23)$$

where again $z = -\square_\Lambda$. Through the above definitions (23), the gauge invariant part of the propagator greatly simplifies to

$$\mathcal{O}^{-1}(k)^{\xi=0} = \frac{1}{k^2} \left(\frac{P^{(2)}}{\bar{h}_2} - \frac{P^{(0)}}{(D-2)\bar{h}_0} \right). \quad (24)$$

We clarify now the incompatibility of the unitarity with a polynomial choice of either \bar{h}_0, \bar{h}_2 or, equally, of the two functions $\alpha(\square), \beta(\square)$. If we assume for a moment $\alpha(\square)$ and $\beta(\square)$ to be polynomial, then each of the two fractions in the propagator takes the following form,

$$\frac{1}{k^2(1+p_n(k^2))} = \frac{c_0}{k^2} + \sum_i \frac{c_i}{k^2 - M_i^2}, \quad (25)$$

where $p_n(x)$ is a polynomial of degree n . In (25) we used the factorization theorem for polynomials and the partial fraction decomposition [48]. When multiplying the left and right side of (25) by k^2 and considering the ultraviolet limit $k^2 \rightarrow +\infty$, we find that at least one of the coefficients c_i is negative. Therefore the theory contains at least a ghost in the spectrum. The conclusion is that h_2, h_0 cannot be polynomial.

Once established that h_2 and h_0 are not polynomial functions, we demand the following general properties for the transcendental entire functions $h_i(z)$ ($i = 0, 2$) and/or $\bar{h}_i(z)$ ($i = 0, 2$) [48]:

- (i) $\bar{h}_i(z)$ ($i = 0, 2$) is real and positive on the real axis and it has no zeroes on the whole complex plane $|z| < +\infty$. This requirement implies that there are no gauge-invariant poles other than the transverse massless physical graviton pole.
- (ii) $|h_i(z)|$ has the same asymptotic behavior along the real axis at $\pm\infty$.
- (iii) There exists $\Theta > 0$ such that

$$\begin{aligned}\lim_{|z| \rightarrow +\infty} |h_i(z)| &\rightarrow |z|^{\gamma+N}, \\ \gamma &\geq D/2 \quad \text{for } D = D_{\text{even}}, \\ \gamma &\geq (D-1)/2 \quad \text{for } D = D_{\text{odd}},\end{aligned}\quad (26)$$

for the argument of z in the following conical regions

$$C = \{z \mid -\Theta < \arg z < +\Theta, \pi - \Theta < \arg z < \pi + \Theta\},$$

for $0 < \Theta < \pi/2$.

This condition is necessary in order to achieve the (supe-)renormalizability of the theory that we are going to show here below. The necessary asymptotic behavior is imposed not only on the real axis, (ii) but also in the conic regions that surround it. In an Euclidean spacetime, the condition (ii) is not strictly necessary if (iii) applies.

Let us then examine the ultraviolet behavior of the quantum theory. According the property (iii) in the high energy regime, the propagator in the momentum space goes as

$$\mathcal{O}^{-1}(k) \sim 1/k^{2\gamma+2N+4} \quad \text{for large } k^2$$

(see (15, 23, 24)). However, the n -graviton interaction has the same leading scaling of the kinetic term, since it can be written in the following schematic way,

$$\begin{aligned}\mathcal{L}^{(n)} &\sim h^n \square_\eta h \, h_i(-\square_\Lambda) \square_\eta h \\ &\rightarrow h^n \square_\eta h (\square_\eta + h^m \partial h \partial)^{\gamma+N} \square_\eta h,\end{aligned}\quad (27)$$

where the indexes for the graviton fluctuation $h_{\mu\nu}$ are omitted and $h_i(-\square_\Lambda)$ is the entire function defined by the properties (i)-(iii). From (27), the superficial degree of divergence in a spacetime of “even” dimension is

$$\begin{aligned}\delta_{\text{even}} &= D_{\text{even}} L - (2\gamma + 2N + 4)I + (2\gamma + 2N + 4)V \\ &= D_{\text{even}} L - (2\gamma + D_{\text{even}})I + (2\gamma + D_{\text{even}})V \\ &= D_{\text{even}} - 2\gamma(L - 1).\end{aligned}\quad (28)$$

On the other hand, in a spacetime of “odd” dimension the upper limit to the degree of divergence is

$$\delta_{\text{odd}} = D_{\text{odd}} - (2\gamma + 1)(L - 1). \quad (29)$$

In (28) and (29) we used again the topological relation between vertexes V , internal lines I and number of loops L : $I = V + L - 1$. Thus, if $\gamma > D_{\text{even}}/2$ or $\gamma > (D_{\text{odd}} - 1)/2$, only 1-loop divergences exist and the theory is super-renormalizable¹. Only a finite number of constants are renormalized in the action (15), i.e. $\kappa, \bar{\lambda}, a_n, b_n$ and the finite number of couplings that multiply the operators in the last line. The renormalized action reads

$$\begin{aligned}S &= \int d^D x \sqrt{|g|} \left[2 Z_\kappa \kappa^{-2} R + Z_{\bar{\lambda}} \bar{\lambda} \right. \\ &\quad + \sum_{n=0}^N \left(Z_{a_n} a_n R (-\square_\Lambda)^n R + Z_{b_n} b_n R_{\mu\nu} (-\square_\Lambda)^n R^{\mu\nu} \right) \\ &\quad + R h_0(-\square_\Lambda) R + R_{\mu\nu} h_2(-\square_\Lambda) R^{\mu\nu} \\ &\quad \left. + Z_{c_1^{(1)}} c_1^{(1)} R^3 + \dots + Z_{c_1^{(N)}} c_1^{(N)} R^{N+2} \right].\end{aligned}\quad (30)$$

¹ A “local” super-renormalizable quantum gravity with a large number of metric derivatives was for the first time introduced in [42].

All the couplings in (30) must be understood as renormalized at an energy scale μ . Contrarily, the functions $h_i(z)$ are not renormalized. To understand this point thoroughly, we can write the generic entire functions as a series, i.e. $h_i(z) = \sum_{r=0}^{+\infty} a_r z^r$. Because of the superficial degrees of divergence (28) and (29), there are no counterterms that renormalize a_r for $r > N$. As a matter of fact, the couplings in the second line of (30) already incorporate the renormalizations of the coefficients a_r for $r \leq N$. Therefore, the non-trivial dependence of the entire functions $h_i(z)$ on their argument is preserved at quantum level.

Imposing the conditions (i)-(iii) we have the freedom to choose the following form for the functions h_i ,

$$h_2(z) = \frac{V(z)^{-1} - 1 - \frac{\kappa^2 \Lambda^2}{2} z \sum_{n=0}^N \tilde{b}_n z^n}{\frac{\kappa^2 \Lambda^2}{2} z},$$

$$h_0(z) = -\frac{V(z)^{-1} - 1 + \kappa^2 \Lambda^2 z \sum_{n=0}^N \tilde{a}_n z^n}{\kappa^2 \Lambda^2 z}, \quad (31)$$

for general parameters \tilde{a}_n and \tilde{b}_n . Here $V(z)^{-1} = e^{H(z)}$ and $H(z)$ is an entire function that exhibits logarithmic asymptotic behavior in the conical region C . The form factor $\exp H(z)$ has no zeros in the entire complex plane for $|z| < +\infty$. Furthermore, the non-locality in the action is actually a “soft” form of non locality, because a Taylor expansion of $h_i(z)$ eliminates the denominator \square_Λ at any energy scale.

The entire function $H(z)$, which is compatible with the property (iii), can be defined as

$$H(z) = \int_0^{p_{\gamma+N+1}(z)} \frac{1 - \zeta(\omega)}{\omega} d\omega, \quad (32)$$

where $p_{\gamma+N+1}(z)$ and $\zeta(z)$ must satisfy the following requirements:

- a. $p_{\gamma+N+1}(z)$ is a real polynomial of degree $\gamma + N + 1$ with $p_{\gamma+N+1}(0) = 0$,
- b. $\zeta(z)$ is an entire and real function on the real axis with $\zeta(0) = 1$,
- c. $|\zeta(z)| \rightarrow 0$ for $|z| \rightarrow \infty$ in the conical region C defined in (iii).

Let us investigate the unitarity of the theory. We assume that the theory is renormalized at some scale μ_0 . If we set

$$\tilde{a}_n = a_n(\mu_0), \quad \tilde{b}_n = b_n(\mu_0), \quad (33)$$

the bare propagator does not possess other gauge-invariant pole than the physical graviton one. Since the energy scale μ_0 is taken as the renormalization point, we get $\bar{h}_2 = \bar{h}_0 = V(z)^{-1} = \exp H(z)$. Thus, only the physical massless spin-2 graviton pole occurs in the bare propagator and (24) reads

$$\mathcal{O}^{-1}(k)^{\xi=0} = \frac{V(k^2/\Lambda^2)}{k^2} \left(P^{(2)} - \frac{P^{(0)}}{D-2} \right). \quad (34)$$

The momentum or energy scale at which the relation between the quantity computed and the quantity measured is identified is called the subtraction point and is indicated usually by “ μ ” [50]. The subtraction point is arbitrary and unphysical, so the final answers do not have to depend on the subtraction scale μ . Therefore, the derivative $d/d\mu^2$ of physical quantities has to be zero. In our case, if we choose another renormalization scale μ , then the bare propagator acquires poles. However, these poles cancel in the dressed physical propagator because the shift in the bare part is cancelled by a corresponding shift in the self energy. The renormalized action (30) will produce finite Green’s functions to whatever order in the coupling constants we have renormalized the theory to. For example, the 2-point Green’s function at the first order in the couplings a_n, b_n can be schematically written as

$$G_{2R}^{-1} = V(k^2/\Lambda^2) (k^2 - \Sigma_R(k^2)), \quad (35)$$

where the renormalization prescription requires that Σ_R satisfies (on shell)

$$\Sigma_R(0) = 0 \quad \text{and} \quad \left. \frac{\partial \Sigma_R}{\partial k^2} \right|_{k^2=0} = 0. \quad (36)$$

We did not consider the tensorial structure and the longitudinal components because they project away when attached to a conserved energy tensor.

The subtraction point is arbitrary and therefore we can take the renormalization prescription off shell to $k^2 = \mu^2$. The couplings we wish to renormalize must be dependent on the chosen subtraction point, $a_n(\mu)$ and $b_n(\mu)$, in such a way that the experimentally measured couplings do not vary on shell. The renormalized Green’s function G_{2R}^{-1} at μ^2 must produce the same Green function when expressed in terms of bare quantities. Consequently, the scalings Z_{a_n} and Z_{b_n} must also depend on μ^2 . The Green’s function written in terms of bare quantities can not depend on μ^2 , but since μ^2 is arbitrary, the renormalized Green’s function must not either. This fact,

$$\frac{dG_{2R}^{-1}}{d\mu^2} = 0, \quad (37)$$

is well known as the renormalization group invariance.

When $h_2(z) = h_0(z) = 0$, the action in (15) reduces to (8) and fails to be unitary. Unitarity can be achieved only if

$$a_n = b_n = \dots = d_n = \dots = 0 \quad \text{in (8)},$$

which corresponds to the D-dimensional Einstein-Hilbert non renormalizable action.

An explicit example of $\exp H(z)$ that satisfies the properties (i)-(iii) can be easily constructed. There are of course many ways to choose $\zeta(z)$, but we focus here on the exponential choice $\zeta(z) = \exp(-z^2)$, which satisfies requirement c. outlined for (32) in a conical region C

with $\Theta = \pi/4$. The entire function $H(z)$ is the result of the integral defined in (32)

$$H(z) = \frac{1}{2} [\gamma_E + \Gamma(0, p_{\gamma+N+1}^2(z))] + \log(p_{\gamma+N+1}(z)),$$

$$\text{Re}(p_{\gamma+N+1}^2(z)) > 0, \quad (38)$$

where $\gamma_E = 0.577216$ is the Euler's constant and $\Gamma(a, z) = \int_z^{+\infty} t^{a-1} e^{-t} dt$ is the incomplete gamma function. If we choose $p_{\gamma+N+1}(z) = z^{\gamma+N+1}$, $H(z)$ simplifies to:

$$H(z) = \frac{1}{2} [\gamma_E + \Gamma(0, z^{2\gamma+2N+2})] + \log(z^{\gamma+N+1}),$$

$$\text{Re}(z^{2\gamma+2N+2}) > 0. \quad (39)$$

Another equivalent expression for the entire function $H(z)$ is given by the following series

$$H(z) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{p_{\gamma+N+1}(z)^{2n}}{2n n!},$$

$$\text{Re}(p_{\gamma+N+1}^2(z)) > 0. \quad (40)$$

For $p_{\gamma+N+1}(z) = z^{\gamma+N+1}$ the Θ angle, which defines the cone C , is $\Theta = \pi/(4\gamma + 4N + 4)$. According to the above expression (40) we find the following behavior near $z = 0$ for the particular choice $p_{\gamma+N+1}(z) = z^{\gamma+N+1}$,

$$H(z) = \frac{z^{2\gamma+2N+2}}{2} - \frac{z^{4\gamma+4N+4}}{8} + \dots \quad (41)$$

We now present a systematic study of the tree-level unitarity [43]. A general theory is well defined if “tachyons” and “ghosts” are absent, in which case the corresponding propagator has only first poles at $k^2 - M^2 = 0$ with real masses (no tachyons) and with positive residues (no ghosts). Therefore, to test the tree-level unitarity of a multidimensional super-renormalizable higher derivative gravity we couple the propagator to external conserved stress-energy tensors, $T^{\mu\nu}$, and we examine the amplitude at the pole. When we introduce a general source, the linearized action including the gauge-fixing reads

$$\mathcal{L}_{hT} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu, \rho\sigma} h^{\rho\sigma} - g h_{\mu\nu} T^{\mu\nu}. \quad (42)$$

The transition amplitude in momentum space is

$$\mathcal{A} = g^2 T^{\mu\nu} \mathcal{O}_{\mu\nu, \rho\sigma}^{-1} T^{\rho\sigma}, \quad (43)$$

where g is an effective coupling constant. Here, only the projectors $P^{(2)}$ and $P^{(0)}$ will give a non zero contribution to the amplitude since the energy tensor is conserved. To make the analysis more explicit, we can expand the

sources using the following set of independent vectors in the momentum space [43],

$$k^\mu = (k^0, \vec{k}), \quad \tilde{k}^\mu = (k^0, -\vec{k}),$$

$$\epsilon_i^\mu = (0, \vec{\epsilon}), \quad i = 1, \dots, D-2, \quad (44)$$

where $\vec{\epsilon}_i$ are unit vectors orthogonal to each other and to \vec{k} . The symmetric stress-energy tensor reads

$$T^{\mu\nu} = a k^\mu k^\nu + b \tilde{k}^\mu \tilde{k}^\nu + c^{ij} \epsilon_i^{(\mu} \epsilon_j^{\nu)}$$

$$+ d k^{(\mu} \tilde{k}^{\nu)} + e^i k^{(\mu} \epsilon_i^{\nu)} + f^i \tilde{k}^{(\mu} \epsilon_i^{\nu)}. \quad (45)$$

The conditions $k_\mu T^{\mu\nu} = 0$ and $k_\mu k_\nu T^{\mu\nu} = 0$ place constraints on the coefficients a, b, d, e^i, f^i [43].

Introducing the spin-projectors (22) and the conservation of the stress-energy tensor $k_\mu T^{\mu\nu} = 0$ in (43), the amplitude results

$$\mathcal{A} = g^2 \left\{ T_{\mu\nu} T^{\mu\nu} - \frac{T^2}{D-2} \right\} \frac{e^{-H(k^2/\Lambda^2)}}{k^2}. \quad (46)$$

Clearly, there is only the graviton pole in $k^2 = 0$ and the residue at $k^2 = 0$ is

$$\text{Res}(\mathcal{A})|_{k^2=0} = g^2 \left[(c^{ij})^2 - \frac{(c^{ii})^2}{D-2} \right] |_{k^2=0}. \quad (47)$$

For $D > 3$ the result above result tells us that $\text{Res}(\mathcal{A})|_{k^2=0} > 0$, which means that the theory is unitary. Instead, for $D = 3$ the graviton is not a dynamical degree of freedom and the amplitude is zero.

A first example of this quantum transition is the interaction of two static point particles. In this case, $T_\nu^\mu = \text{diag}(\rho, 0, 0, 0)$ with $\rho = M \delta(\vec{x})$ and the amplitude (46) simplifies to

$$\mathcal{A} = g^2 \rho^2 \left(\frac{D-3}{D-2} \right) \frac{e^{-H(k^2/\Lambda^2)}}{k^2}, \quad (48)$$

which is positive in $D > 3$ and zero for $D = 3$ since, again, there are no local degrees of freedom in $D = 3$.

A second example we want to consider is the light bending in the multidimensional nonlocal gravity. We consider a static source and a light ray. The amplitude for this process is

$$\mathcal{A} = g^2 T^{\mu\nu} \mathcal{O}_{\mu\nu, \rho\sigma}^{-1} T_{\text{EM}}^{\rho\sigma}, \quad (49)$$

where $T^{\mu\nu}$ is the above energy tensor for the static particle and $T_{\text{EM}}^{\mu\nu}$ is the traceless electro-magnetic energy tensor. Using the projectors defined in (22) and the propagator (34), we obtain

$$\mathcal{A} = g^2 T^{00} T_{\text{EM}}^{00} \frac{e^{-H(k^2/\Lambda^2)}}{k^2}. \quad (50)$$

We see that, at low energy ($\Lambda \rightarrow +\infty$), the amplitude (50) is precisely the amplitude for the interaction between a static source and a light ray in D -dimensional linearized

Einstein's gravity. On the other hand, at high energy the light bending is much smaller in the nonlocal theory than in the Einstein's one.

The proposed theory is not unique, but all the freedom present in the action can be read in the entire function $V(z)$ or $H(z)$ [27, 32, 45–49]. The expression “form factor” for the function $V(z)$ used throughout the paper is not accidental. Indeed, It may be read in analogy with the form factors present in the interaction between a photon and a nucleon. Most importantly, in view of this, it can also be measured experimentally. Specifically, the quantities (48) and (50), at least in principle, can be measured experimentally to fit $V(z)$. The four graviton amplitude is another measurable quantity of the form factor $V(z)$ and it reads

$$\mathcal{A}_{4\text{-grav.}} = \mathcal{A}_{4\text{-grav.}}(s, t, u; V(s, t, u); \epsilon_{1,2,3,4}), \quad (51)$$

where $\epsilon_{1,2,3,4}$ are the four gravitons polarizations and s, t, u the Mandelstam variables. Since $V(z)$ has to be an entire function, we can falsify the theory by comparing the experimental four-gravitons amplitude with (51).

To address the problem of classical singularities mentioned at the beginning of the paper, we can start out by calculating the gravitational potential. Given the modified propagator (34), the graviton solution of the equations of motion resulting from the Lagrangian (42) with $g = \kappa/2$, is

$$\begin{aligned} h_{\mu\nu}(x) &= \frac{\kappa}{2} \int d^D x' \mathcal{O}_{\mu\nu,\rho\sigma}^{-1}(x - x') T^{\rho\sigma}(x') \\ &= \frac{\kappa}{2} \int d^D x' \int \frac{d^D k}{(2\pi)^D} e^{ik(x-x')} \\ &\quad \frac{e^{-H(k^2/\Lambda^2)}}{k^2} \left(T_{\mu\nu} - \frac{\eta_{\mu\nu}}{D-2} T \right). \end{aligned} \quad (52)$$

For a static source with energy tensor

$$T_{\nu}^{\mu} = \text{diag}(M \delta^{D-1}(\vec{x}), 0, \dots, 0), \quad (53)$$

the solution of (52) satisfying spherical symmetry reads

$$\begin{aligned} h_{\mu\nu}(r) &= -\frac{\kappa M}{2} E_{\mu\nu} \int \frac{d^{D-1} k}{(2\pi)^{D-1}} \frac{e^{-i\vec{k}\cdot\vec{x}} e^{-H(\vec{k}^2/\Lambda^2)}}{k^2} \\ &= -\frac{\kappa M}{2} \frac{\pi^{\frac{D-3}{2}}}{(2\pi)^{D-2}} \frac{E_{\mu\nu}}{r^{D-3}} \times \\ &\quad \times \int dp p^{D-4} e^{-H(p^2/r^2\Lambda^2)} {}_0\tilde{F}_1\left(\frac{D-1}{2}; -\frac{p^2}{4}\right), \end{aligned} \quad (54)$$

where ${}_0\tilde{F}_1(a; z) = {}_0F_1(a; z)/\Gamma(a)$ is the regularized hypergeometric confluent function. In (54), we also have introduced the variable $p = |\vec{k}|r$ and the matrix

$$E_{\mu\nu} = \text{diag}\left(\frac{D-3}{D-2}, \frac{1}{(D-2)}, \dots, \frac{1}{D-2}\right).$$

For $r \rightarrow 0$, the entire function $H(z) \approx \log z^{\gamma+N+1}$ and the solution (54) is approximated by

$$\begin{aligned} h_{\mu\nu}(r) &\approx -\frac{\kappa M}{2} \frac{\pi^{\frac{D-3}{2}}}{(2\pi)^{D-2}} E_{\mu\nu} \Lambda^{2\gamma+2N+2} r^{2\gamma+2N+5-D} \times \\ &\quad \times \int dp p^{D-(2N+4)-2\gamma-2} {}_0\tilde{F}_1\left(\frac{D-2}{2}; -\frac{p^2}{4}\right). \end{aligned} \quad (55)$$

The solution (55) is clearly regular near $r \approx 0$ since the exponent of the radial coordinate is always positive in any dimension D .

The gravitational potential is related to the h_{00} component of the graviton field by $\Phi = \kappa h_{00}/2$. Then, using (54) we get

$$\begin{aligned} \Phi(r) &= -\frac{\kappa^2 M}{4} \frac{D-3}{D-2} \int \frac{d^{D-1} k}{(2\pi)^{D-1}} \frac{e^{-H(\vec{k}^2/\Lambda^2)} e^{-i\vec{k}\cdot\vec{x}}}{k^2} \\ &= -\frac{G_N M}{r^{D-3}} 2 \frac{D-3}{D-2} \frac{\Gamma(\frac{D-3}{2})}{\pi^{\frac{D-3}{2}}} \mathbb{F}_D(r), \end{aligned} \quad (56)$$

$$\mathbb{F}_D(r) \equiv \frac{2^{4-D}}{\Gamma(\frac{D-3}{2})} \int dp \frac{e^{-H(\frac{p^2}{r^2\Lambda^2})}}{p^{4-D}} {}_0\tilde{F}_1\left(\frac{D-1}{2}; -\frac{p^2}{4}\right).$$

For example, in $D = 4$, (56) simplifies to

$$\Phi(r) = -\frac{G_N M}{r} \underbrace{\frac{2}{\pi} \int_0^{+\infty} dp J_0(p) e^{-H(p^2/r^2\Lambda^2)}}_{\mathbb{F}(r)}, \quad (57)$$

and (57) can be integrated numerically. In this latter, $J_0(p) = \text{sinc}(p) \equiv \sin(p)/p$ is the Bessel function. For small values of the radial coordinate “ r ” (large values of “ p ”) we get

$$\Phi \approx -2G_N M (\text{const.}) \Lambda^{2\gamma+2} r^{2\gamma+1}, \quad (58)$$

where $\text{const.} \approx 3 \times 10^7 \pi$ for $\Lambda = 1$ and $G_N = 1$. The potential (58) is regular for $r \rightarrow 0$ and a plot of the exact potential (57) for $\gamma = 3$ and $M = 10$ is given in Fig.1.

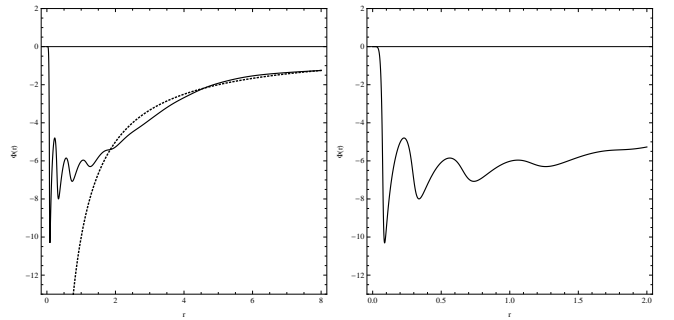


FIG. 1: Plot of the gravitational potential in $D = 4$ for $\gamma = 3$ and $M = 10$ ($\Lambda = G_N = 1$).

Making use of the gravitational potential (57), for the sake of simplicity we can study the “Newtonian cosmology” in $D = 4$. To derive the Friedmann equation in

Newtonian cosmology, we need the kinetic and potential energy of a test particle and we must implement energy conservation [51, 52]. We now consider an observer in a uniform expanding medium of mass density ρ . Because the Universe is homogeneous and isotropic, we can assume any point to be its center. We then identify a particle of mass m at a radial distance r . Due to Newton's theorem, the particle only feels a force from the material at smaller values of r . The material has mass $M = 4\pi\rho r^3/3$ and the constant total energy of the test particle is $E = T + U$, where $T = m\dot{r}^2/2$, $U = m\Phi(r)$ and $\Phi(r)$ is given in (57). Because the Universe is homogeneous, we apply this argument to any couple of particles, which allows us to introduce comoving coordinates defined by $\vec{r} = a(t)\vec{x}$. For the same reason, the real distance \vec{r} is related to the comoving distance \vec{x} by $a(t)$, which is a function of time alone. When dividing $E = T + U$ by $a(t)^2\vec{x}^2$, we get the modified Friedmann equation [53, 54],

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho\mathbb{F}(a) - \frac{K}{a^2}, \quad (59)$$

where $K = -2E/m\vec{x}^2$ and $\mathbb{F}(a)$ is defined in (57). To maintain homogeneity, the quantity E must depend on the comoving coordinates according to $E \propto \vec{x}^2$. For the same reason, we have rescaled $\Lambda^2 \propto 1/\vec{x}^2$ obtaining an equation independent from \vec{x} so that homogeneity is respected. Hereon we assume $K = 0$. As we know the scaling of the gravitational potential for small values of “ r ”, we then can get the Friedmann equation near $a(t) \approx 0$. In this limit we find

$$H^2 = \frac{8\pi G_N}{3}\rho \frac{2(\text{const.})}{\pi} a^{2\gamma+2}, \quad (60)$$

where the constant is defined right after (58). It is clear that this Universe is singularity-free, since $H \rightarrow 0$ when $a(t) \rightarrow 0$. Furthermore, if we add the cosmological constant Λ_{cc} to (59), then the equation (60) reads

$$H^2 = \frac{8\pi G_N}{3}\rho \frac{2(\text{const.})}{\pi} a^{2\gamma+2} + \frac{\Lambda_{cc}}{3}. \quad (61)$$

This shows that the cosmological constant dominates the Universe at high energy whatever kind of matter we introduce, either dust or radiation. Consequently, the Universe follows a natural de Sitter evolution at the Planck scale. We can numerically integrate equation (59) as Fig.2 shows for both dust matter and radiation.

In a D -dimensional spacetime, the Newtonian cosmology can be theorized in a way similar to the 4-dimensional case. The Friedmann equation (for $K = 0$) together with the fluid one, reads

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{16\pi G_N}{(D-2)(D-1)}\rho\mathbb{F}_D(a),$$

$$\dot{\rho} + (D-1)\left(\frac{\dot{a}}{a}\right)(\rho + P) = 0 \quad (\text{fluid eq.}), \quad (62)$$

with $\mathbb{F}_D(a)$ as defined in (56). The fluid equation in (62) has been obtained using the first law of thermodynamics

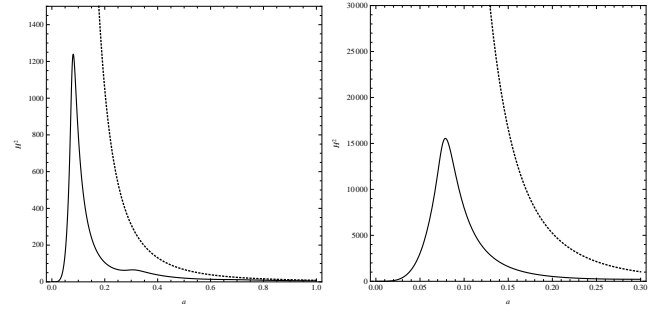


FIG. 2: Plot of $H^2(a)$ in $D = 4$ for $\gamma = 3$ and $\Lambda = G_N = 1$ in a universe dominated by dust matter on the left and radiation on the right. The dashed lines account for the classical trajectories.

$dE + P dV = T dS$ (P is the pressure, T the temperature and S the entropy) and assuming reversible expansion ($dS = 0$). In D dimensions, the state equation for dust matter is $P = 0$ and for radiation is $\rho = (D-1)P$, so that the fluid equation implies

$$\rho_{\text{dust}} = \frac{\rho_{\text{dust}}^0}{a^{D-1}}, \quad \rho_{\text{rad.}} = \frac{\rho_{\text{rad.}}^0}{a^D}. \quad (63)$$

When $a \rightarrow 0$, the function $\mathbb{F}_D(a) \approx (a\Lambda)^{2\gamma+2N+2}$ and the Friedmann equation simplifies to

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 \approx \frac{16\pi G_N}{(D-2)(D-1)}\rho(a\Lambda)^{2\gamma+2N+2}. \quad (64)$$

In (64) H^2 goes to zero for both radiation and dust matter, which means that the D -dimensional Newtonian cosmos are singularity-free.

Consistently with the singularity-free cosmology we have illustrated so far, the black hole solutions turn out to be regular, as we are going to demonstrate in $D = 4$.

Following [55–62], the equations of motion for the above theory (up to square curvature terms) are

$$G_{\mu\nu} + O(R_{\mu\nu}^2) + O(\nabla^2 R_{\rho\sigma}) = 8\pi G_N V(z) T_{\mu\nu}, \quad (65)$$

where the argument $z = -\square_\Lambda$ as defined throughout in the paper².

Since we are going to solve the Einstein equations neglecting curvature square terms, then we have to impose the conservation $\nabla^\mu (V(z) T_{\mu\nu}) = 0$ in order for the theory to be compatible with the Bianchi identities. Conversely,

² In general, a differential equation with an infinity number of derivative has not a well-defined initial value problem and it needs an infinite number of initial conditions. It is shown in [66–68] that in a general framework each pole of the propagator contributes two initial data to the final solution. This is precisely our case because the only pole in the bare propagator is the massless graviton and the theory has a well defined Cauchy problem.

the exact equations of motion satisfy the Bianchi identities because the theory presents general covariance. The condition $\nabla^\mu (V(z) T_{\mu\nu}) = 0$ compensates the truncation in the modified Einstein equations (65).

Our main purpose is to solve the field equations by assuming a static source, which means that the four-velocity field u^μ has only a non-vanishing time-like component $u^\mu \equiv (u^0, \vec{0})$, $u^0 = (g^{00})^{-1/2}$ [79–98]. We consider the component T^0_0 of the energy-momentum tensor for a static source of mass M (53). In polar coordinates, $T^0_0 = \rho = M\delta(r)/4\pi r^2$ [65]. The metric of our space-time is assumed to be given by the usual static, spherically symmetric Schwarzschild form

$$ds^2 = F(r)dt^2 - \frac{dr^2}{F(r)} - r^2\Omega^2, \quad (66)$$

$$F(r) = 1 - \frac{2G_N m(r)}{r}.$$

The effective energy density and pressures are defined by

$$V(z)T^\mu_\nu = \frac{G^\mu_\nu}{8\pi G_N} = \text{Diag}(\rho^e, -P_r^e, -P_\perp^e, -P_\perp^e). \quad (67)$$

We temporarily adopt free-falling Cartesian-like coordinates [64, 65] to calculate the effective energy density, assuming $p_{\gamma+N+1}(z) = z^4$ in (38),

$$\begin{aligned} \rho^e(\vec{x}) &:= V(-\square_\Lambda)T^0_0 = M V(-\square_\Lambda) \delta(\vec{x}) \\ &= M \int \frac{d^3k}{(2\pi)^3} e^{-H(\vec{k}^2/\Lambda^2)} e^{i\vec{k}\cdot\vec{x}} \\ &= \frac{2M}{(2\pi)^2 r^3} \int_0^{+\infty} e^{-H(p^2/r^2\Lambda^2)} p \sin(p) dp, \end{aligned} \quad (68)$$

where $r = |\vec{x}|$ is the radial coordinate. Here we introduced the Fourier-transform for the Dirac delta function and we also introduced a new dimensionless variable in the momentum space, $p = |\vec{k}|r$, where \vec{k} is the physical momentum. The energy density distribution defined in (68) respects spherical symmetry. We evaluated numerically the integral in (68) and the resulting energy density is plotted in Fig.3. In the low energy limit we can expand $H(z)$ for $z = -\square/\Lambda^2 \ll 1$ and we can integrate analytically (68)

$$\rho^e(r) = \frac{2M}{(2\pi)^2 r^3} \int_0^{+\infty} e^{-p^{16}/(2r^{16}\Lambda^{16})} p \sin(p) dp. \quad (69)$$

The result is extremely complex, and its plot is given in Fig.3; however, the Taylor expansion near $r \approx 0$ generates a constant leading order $\rho^e(r) \propto M\Lambda^3$.

The covariant conservation and the additional condition, $g_{00} = -g_{rr}^{-1}$ fully specify the form of $V(z)T^\mu_\nu$ and the Einstein's equations reads

$$\begin{aligned} \frac{dm(r)}{dr} &= 4\pi\rho^e r^2, \\ \frac{1}{F} \frac{dF}{dr} &= \frac{2G_N (m(r) + 4\pi P_r^e r^3)}{r(r - 2G_N m(r))}, \\ \frac{dP_r^e}{dr} &= -\frac{1}{2F} \frac{dF}{dr} (\rho^e + P_r^e) + \frac{2}{r} (P_\perp^e - P_r^e). \end{aligned} \quad (70)$$

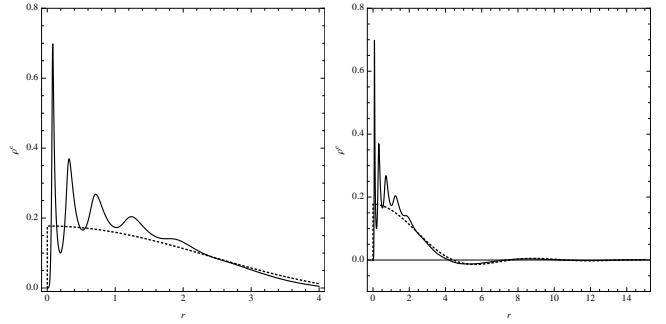


FIG. 3: Plot of the energy density for $m = 10$ in Planck units assuming $\Lambda = m_P$. The solid line is a plot of (68) without any approximation. The dashed line refers to the energy density profile (69) in the limit $-\square/\Lambda^2 \ll 1$.

Because of the complicated energy density profile, this is how the first Einstein equation would fit in (70)

$$m(r) = 4\pi \int_0^r dr' r'^2 \rho^e(r'). \quad (71)$$

However, the energy density goes to zero at infinity, reproducing the asymptotic Schwarzschild spacetime with $m(r) \approx M$ (constant). On the other hand, it is easy to calculate the energy density profile close to $r \approx 0$ since $H(z) \rightarrow \log z^4$ for $z \rightarrow +\infty$ (or $r \rightarrow 0$ in (68)). In this regime, $m(r) \propto M\Lambda^8 r^8$ and, for a more general monomial $p_{\gamma+1}(z) = z^{\gamma+1}$, $m(r) \propto M(\Lambda r)^{2\gamma+2}$. The function $F(r)$ in (66) near to $r \approx 0$ is approximated by

$$F(r) \approx 1 - c G_N M \Lambda^{2\gamma+2} r^{2\gamma+1}, \quad (72)$$

where c is a dimensionless constant.

We show now that the metric has at least two horizons: an event horizon and a Cauchy horizon. The metric interpolates two asymptotic flat regions, one at infinity and the other in $r = 0$, so that we can write the $g_{rr}^{-1} = -F$ component in the following way

$$F(r) = 1 - \frac{2MG(r)}{r}. \quad (73)$$

Here $G(r) \rightarrow G_N$ for $r \rightarrow \infty$, $G(r) \propto G_N r^{2\gamma+2}$ for $r \rightarrow 0$ and $G(r)$ does not depend on the mass M . The function $F(r)$ goes to “1” in both limits (for $r \rightarrow +\infty$ and $r \rightarrow 0$). Since M is a multiplicative constant, we can always vary it for a fixed value of the radial coordinate r , such that $F(r)$ becomes negative. From this it follows that the function $F(r)$ must change sign at least twice. The second equation in (70) is solved by $P_r^e = -\rho^e$ and the third one defines the transversal pressure once the energy density ρ^e is known. For the lapse function $F(r)$ in (72), we can calculate the Ricci scalar and the Kretschmann invariant

$$\begin{aligned} R &= c G_N M \Lambda^{2\gamma+2} (2\gamma+2)(2\gamma+3) r^{2\gamma-1}, \\ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} &= \\ &= 4c^2 G_N^2 M^2 \Lambda^{4\gamma+4} (4\gamma^4 + 4\gamma^3 + 5\gamma^2 + 4\gamma + 2) r^{4\gamma-2}. \end{aligned} \quad (74)$$

By evaluating the above curvature tensors at the origin one finds that they are finite for $\gamma > 1/2$ and, in particular, for the minimal super-renormalizable theory in $D = 4$ with $\gamma \geq 3$.

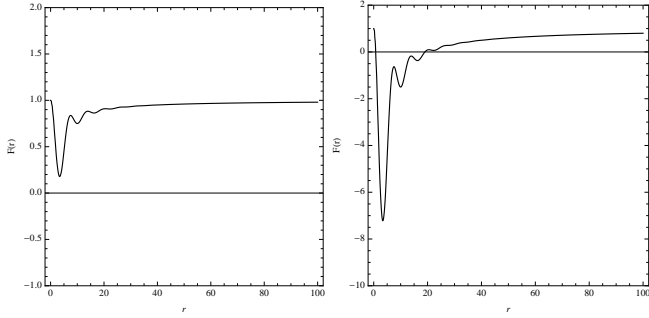


FIG. 4: The first plot shows the function $F(r)$ for the energy profile (69) and $H(z)$ defined in (39) with the parameter $\gamma = 3$. The ADM mass values are $M = 1$ and $M = 10$ (in Planck units) for the first and the second plot respectively.

The form factor $V(z)$ is able to tame the curvature singularity of the Schwarzschild solution at least for the truncation of the theory here analyzed. However, we believe that the higher order corrections to the Einstein's equations will not substantially change the fundamental remarkable feature of the solutions found in this section [69].

Besides, we can exactly (but only numerically) integrate the modified Einstein equations (65) for the energy density defined in (68). Using the integral form of the mass function (71), we achieve the metric component $F(r)$ defined in (66). The numerical results are plotted in Fig.4 for different values of the ADM mass M . The metric function $F(r)$ can intersect no times, twice or more than twice the horizontal axis according to the value of the ADM mass M . This may enable “multi-horizon black holes” as an exact solution of the modified equations of motion (65).

We expect the same features to be maintained in any dimension $D > 4$.

We can now address a more general class of theories following the Efimov's book on non local interactions [27]. Let us consider the propagator in the following general form

$$\mathcal{O}^{-1}(z) = \frac{V(z)}{z \Lambda^2} \quad (75)$$

(the notation is rather compatible with the graviton propagator (34) and $z := -\square_\Lambda$).

As was shown by Efimov [27], the nonlocal field theory is “unitary” and “microcausal” provided that the following properties are satisfied by $V(z)$,

- I. $V(z)$ is an entire analytic function in the complex z -plane and it has a finite order of growth $1/2 \leq \rho < +\infty$ i.e. $\exists b > 0, c > 0$ so that

$$|V(z)| \leq c e^{b|z|^\rho}. \quad (76)$$

- II. When $\text{Re}(z) \rightarrow +\infty$ ($k^2 \rightarrow +\infty$), $V(z)$ decreases quite rapidly. For example, we can consider the following cases.

- a. $V(z) = O\left(\frac{1}{|z|^a}\right)$, $a > \frac{D-2}{2}$. For $a = \frac{D-2}{2}$ the theory is not super-renormalizable, but may still be renormalizable.

- b. $\lim_{\text{Re}(z) \rightarrow +\infty} |z|^N |V(z)| = 0, \forall N > 0$.

- III. $[V(z)]^* = V(z^*)$.

- IV. $V(0) = 1$.

- V. The function $V(z)$ can be non-negative on the real axis, i.e. $V(x) \geq 0$.

Here are some examples of possible functions:

- A. $V_A(z) = e^{-z^n}$ for $n \in \mathbb{N}_+$, the weight is $\rho = n < +\infty$,

- B. $V_B(z) = \left(\frac{\sin \sqrt{z}}{\sqrt{z}}\right)^{2a}$,

- C. $V_C(z) = 2^s \Gamma(1+s) \frac{J_s(\sqrt{z})}{(\sqrt{z})^s}$ ($s > 0$).

When $V(z) = V_B(z)$ or $V_C(z)$, the functions $h_i(z)$ in the action are not entire functions, so they do no longer meet our minimal requirement.

A more refined growth measure is obtained by defining the order $\rho(\theta_1, \theta_2)$ for $V(z)$ in the angle $\theta_1 \leq \arg z \leq \theta_2$. It is a remarkable property of entire functions that, for appropriate $V(z)$, $\rho(\theta_1, \theta_2)$ may range from zero to arbitrarily large values as θ_1, θ_2 change. The function $V(z)^{-1} = \exp H(z)$ that we have introduced and extensively studied in the first part of the paper exhibits at most polynomial behavior along the real axis and it is of infinite order $\rho = +\infty$ in the full complex plane.

To expand on the point II.a, we calculate the propagator in the coordinate space for a general form factor $V(z)$. The Fourier transform of (75) reads

$$G(x) = \int \frac{d^D k}{(2\pi)^D} \frac{V(k^2 \ell^2)}{k^2} e^{i k x}, \quad \ell \equiv 1/\Lambda, \quad (77)$$

where we neglected any tensorial structure and we assumed Euclidean signature. Changing the existing coordinates into D -dimensional spherical ones and integrating (77) in the angular variables, we get

$$G(x) = \frac{\pi^{\frac{D-3}{2}}}{(2\pi)^{D-1} \Gamma\left(\frac{D-1}{2}\right)} \int_0^{+\infty} du \frac{u^{\frac{D-4}{2}} V(u \ell^2)}{2} \sqrt{\pi} \Gamma\left(\frac{D-1}{2}\right) {}_0\tilde{F}_1\left(\frac{D}{2}; -\frac{u x^2}{4}\right), \quad (78)$$

where we have introduced the variable $u = k^2$. From II.a, $V(u l^2) = O(1/u^a)$ for $u \rightarrow +\infty$ and since ${}_0\tilde{F}_1 \approx \text{const.}$ for $x^2 \rightarrow 0$, the propagator in the coincidence limit is finite only for certain values of a ,

$$G(0) \propto \int_0^{+\infty} du u^{\frac{D-4}{2}-a} < \infty \iff a > \frac{D-2}{2}. \quad (79)$$

For $D = 4$ the two-point function in the coordinate space is

$$G(x) = \frac{1}{(2\pi)^2} \int_0^{+\infty} du V(u l^2) \frac{J_1(\sqrt{u x^2})}{2\sqrt{u x^2}}, \quad (80)$$

where “ J_1 ” is the Bessel function of the first kind “ $J_n(z)$ ”. Using II.a ($u \rightarrow +\infty$) and/or the short distances limit $x^2 \rightarrow 0$, the propagator (80) reads

$$G(x) = \begin{cases} O\left(\frac{1}{(x^2)^{1-a}}\right) & \text{for } 0 < a < 1, \\ O(\ln(x^2)) & \text{for } a = 1, \\ O(1) & \text{for } a > 1. \end{cases} \quad (81)$$

Only for $a > 1$, $G(0) < +\infty$ in the coincidence limit. This is further evidence that super-renormalizability requires $a > (D-2)/2$. Later on we will show that we may still have renormalizability for $a = (D-2)/2$ in the case study $D = 4$.

We now move to investigate the general theory (15) with form factor $V_A(z) = \exp(-z^n)$, which satisfies the property II.b., in the entire functions $h_i(z)$ defined in (31). The high energy propagator reads

$$\mathcal{O}^{-1}(k) = \frac{e^{-(k^2/\Lambda^2)^n}}{k^2}. \quad (82)$$

The m -graviton interaction has the same scaling, since it can be written in the following schematic way

$$\begin{aligned} \mathcal{L}^{(m)} &\sim h^m \square_\eta h \, h_i(-\square_\Lambda) \square_\eta h \\ &\rightarrow h^m \square_\eta h \frac{e^{(-\frac{\square_\eta}{\Lambda^2})^n}}{\square_\eta} \square_\eta h + \dots, \end{aligned} \quad (83)$$

where $\square_\eta = \eta^{\mu\nu} \partial_\mu \partial_\nu$. The notation “...” indicates other sub-leading interaction terms coming from the covariant D'Alembertian operator. Placing an upper bound to the amplitude with L -loops, we find

$$\begin{aligned} \mathcal{A}^{(L)} &\leq \int (d^D k)^L \left(\frac{e^{-k^{2n}/\Lambda^{2n}}}{k^2} \right)^I \left(e^{k^{2n}/\Lambda^{2n}} k^2 \right)^V \\ &= \int (dk)^{DL} \left(\frac{e^{-k^{2n}/\Lambda^{2n}}}{k^2} \right)^{I-V} \\ &= \int (dk)^{DL} \left(\frac{e^{-k^{2n}/\Lambda^{2n}}}{k^2} \right)^{L-1}. \end{aligned} \quad (84)$$

In the last step we used again the topological identity $I = V + L - 1$. The L -loops amplitude is UV finite

for $L > 1$ and it diverges as “ k^D ” for $L = 1$. Only 1-loop divergences survive in this theory. Therefore, the theory is super-renormalizable and unitary, as well as microcausal as pointed out in [27–32].

To calculate the gravitational potential for $n = 1$, it suffices to replace $\exp H(p^2/r^2 \Lambda^2) \rightarrow \exp(p^2/r^2 \Lambda^2)$ within the integral (54). The result is:

$$\begin{aligned} h_{\mu\nu}(r) &= -\frac{\kappa M}{2} \frac{1}{4\pi^{\frac{D-1}{2}} r^{D-3}} E_{\mu\nu} \times \\ &\times \left[\Gamma\left(\frac{D-3}{2}\right) - \Gamma\left(\frac{D-3}{2}; \frac{r^2 \Lambda^2}{4}\right) \right]. \end{aligned} \quad (85)$$

To prove the regularity of the graviton solution, we expand (85) near $r = 0$, so that we get the following finite leading term

$$h_{\mu\nu}(0) = -E_{\mu\nu} \frac{\kappa M 2^{1-D} \Lambda^{D-3}}{(D-3) \pi^{\frac{D-1}{2}}}. \quad (86)$$

For $D = 4$, (85) simplifies to

$$\Phi(r) = -\frac{G_N M}{r} \text{Er}\left(\frac{r \Lambda}{2}\right). \quad (87)$$

The gravitational potential is regular in $r = 0$ and its value is $\Phi(0) = -G_N M \Lambda / \sqrt{\pi}$. For $n > 1$, the potential is still regular in $r = 0$ and it takes the value $\Phi(0) \propto -G_N M \Lambda$ with a slightly different coefficient.

In the case $n = 1$, we can always solve the equations of motion (65) for a spherically symmetric D -dimensional spacetime with metric

$$ds_D^2 = F_D(r) dt^2 - \frac{dr^2}{F_D(r)} - r^2 d\Omega_{D-2}, \quad (88)$$

where $d\Omega_{D-2}$ is described in terms of $D-2$ angles. The form factor $V(z) = \exp -z$ gives a smearing of the source and the energy density reads

$$\rho^e(r) = V(z) T^0_0 = M \left(\frac{\Lambda^2}{4\pi} \right)^{\frac{D-1}{2}} e^{-r^2 \Lambda^2/4}. \quad (89)$$

Integrating the “00” component of the modified Einstein equations (65), we get the function $F_D(r)$,

$$\begin{aligned} F_D(r) &= 1 - \frac{2MG_D}{\Gamma\left(\frac{D-1}{2}\right) r^{D-3}} \gamma\left(\frac{D-1}{2}; \frac{r^2 \Lambda^2}{4}\right) \\ \gamma\left(\frac{D-1}{2}, \frac{r^2 \Lambda^2}{4}\right) &\equiv \int_0^{r^2 \Lambda^2/4} \frac{dt}{t} t^{\frac{D-1}{2}} e^{-t}, \\ \Gamma((D-1)/2) &= \left[\frac{D-1}{2} - 1 \right]! \text{ for } D \text{ odd}, \\ \Gamma((D-1)/2) &= \sqrt{\pi} \left[\frac{(D-3)!!}{2^{(D-2)/2}} \right] \text{ for } D \text{ even}. \end{aligned} \quad (90)$$

where $[G_D] = M^{2-D}$. The other components in (65) are solved by $\rho^e = -P_r^e$, while the covariant conservation of the effective energy tensor $V(z) T_{\mu\nu}$ determines

$$T^{e i}_i = -\rho^e - (D-2)^{-1} r \partial_r \rho^e, \quad i = 1, \dots, D-2. \quad (91)$$

The metric has a “de Sitter core” near the origin $r = 0$ where

$$F_D(r) \approx 1 - \frac{4MG_D \Lambda^{D-1}}{(D-1)2^{D-2}\pi^{(D-3)/2}} r^2, \quad (92)$$

from which descends a singularity-free spacetime. All the other properties of the metric have been extensively studied in [99].

The repercussions of this study affect several fields as emerges from previous investigations in LHC black hole phenomenology [100, 101], gauge gravity duality [102] and early universe cosmology [62, 97]. Specifically it has been shown that the resulting black hole tend to emit softer particles on the brane [101], a fact which is in marked contrast with previous results based on classical metrics.

Another special theory we wish to explore is defined by the following form factor,

$$\begin{aligned} V(z) &= e^{-H(z)}, \\ H(z) &= \frac{1}{2} [\gamma_E + \Gamma(0, z^{2N+2})] + \log[z^{N+1}], \\ \text{Re}(z^{2N+2}) &> 0. \end{aligned} \quad (93)$$

This form factor has been achieved from (38) by choosing $\gamma = 0$. The theory satisfies all the properties I - V of the second class of theories here examined. In particular, the behavior of the entire functions $h_i(z)$ for $|z| \rightarrow +\infty$ is,

$$\begin{aligned} \lim_{|z| \rightarrow +\infty} |h_i(z)| &\rightarrow |z|^N, \quad \text{for } z \text{ in :} \\ C &= \{z \mid -\Theta < \arg z < +\Theta, \pi - \Theta < \arg z < \pi + \Theta\}, \\ \text{for } \Theta &= \pi/(4N+4). \end{aligned} \quad (94)$$

Since in even dimensions $N = (D_{\text{even}} - 4)/2$, the entire functions $h_i(z)$ in $D = 4$ approach a constant for $|z| \rightarrow +\infty$. This theory embodies the quadratic Stelle action in the ultraviolet limit but without any ghost pole in the propagator. The form factor cross-connects the quadratic action in the infrared with an equivalent theory in the ultraviolet.

The theory in question meets the property II.a for the critical value $a = (D - 2)/2$ ($a = 1$ in $D = 4$) [27]³. The amplitudes are divergent at each order in the loop expansion and the maximal superficial degree of divergence from (7) or (9) is $\delta = D$ as it occurs in the local theory. Therefore, the theory ceases to be super-renormalizable, but it preserves renormalizability and unitarity as it can be inferred from (46) and (47) with the entire function $H(z)$ defined in (93).

The gravitational potential can be obtained integrating (57) with the form factor (93). The potential is regular everywhere and $\Phi(r) \approx -14 G_N M \Lambda^2 r$ near the point $r = 0$. Because the metric scaling $F(r) \approx 1 - (\text{const.}) M \Lambda^2 r$ in $r = 0$, black hole solutions are not singularity-free, as proved by the diverging curvature invariants [105–107].

Let us assume that the coupling constants in $D = 4$ satisfy the following relation (see (30))

$$b_0(Z_{b_0} - 1) = 3a_0(1 - Z_{a_0}) + 1 \quad (95)$$

in the ultraviolet regime. Then the Lagrangian turns out to be conformal invariant at high energy [104],

$$\begin{aligned} S_{\text{UV}}^{D=4} &\propto \int d^4x \sqrt{|g|} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \\ &\equiv 2 \int d^4x \sqrt{|g|} \left(R_{\mu\nu} R^{\mu\nu} - \frac{R^2}{3} \right). \end{aligned} \quad (96)$$

However, the same result can not be achieved for the local Stelle’s theory, because the relationship between the coupling constants, for which we get a conformal invariant action, is the same one by which the theory loses its renormalizability.

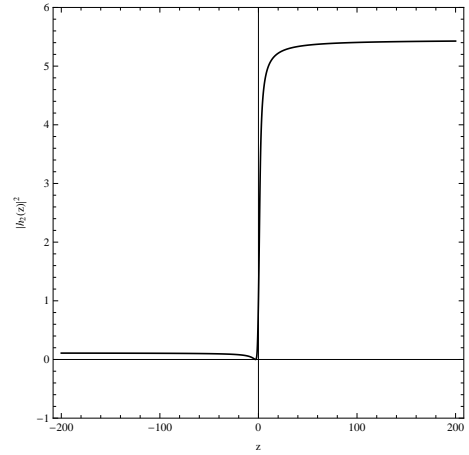


FIG. 5: Plot of $|h_2(z)|^2$ with the form factor defined in (93) for $D = 4$ and then $N = 0$. To draw this plot we have taken $\kappa^2 = 2$, $\Lambda = 1$ and $\tilde{b}_0 = 1$ in (31).

A fundamental quantity that explains “the spacetime structure” is the spectral dimension (hereafter d_s). This is not only a tool to compare different approaches to quantum gravity, but it is actually a device to extract information about the physics within the spacetime. It is equivalent knowing either the spectral dimension, the propagator or the gravitational potential. Schematically,

$$d_s \iff \text{Gravitational Potential}. \quad (97)$$

More explicitly, if we know the spectral dimension, then we also know the heat kernel (see below), from which we

³ See page 147 and pages 246-252 of the Efimov’s study [27].

can derive the propagator and the ensuing gravitational potential. Clearly, the reverse relationship is true as well.

Here we calculate the spacetime spectral dimension flow from short to long distances for the three different cases already discussed in the paper,

Form Factor 1. $V_1(z) = e^{-H(z)}$,

Form Factor 2. $V_2(z) = e^{-z^n}$, $n \in \mathbb{N}_+$,

Form Factor 3. $V_3(z) = e^{\frac{1}{2}[\gamma_E + \Gamma(0, z^{2N+2})] + \log(z^{N+1})}$.

As we are going to show, renormalizability, along with unitarity, implies a spectral dimension $d_s < 1$ for the the form factor 1, $d_s = 0$ for the form factor 2 and $d_s = 2$ for the form factor 3. Let us recall the definition of spectral dimension in quantum gravity. Such definition is borrowed from the theory of diffusion processes on fractals [14] and adapted to the quantum gravity context. In the Brownian motion of a test particle moving on a D -dimensional Riemannian manifold \mathcal{M} with a fixed smooth metric $g_{\mu\nu}^o(x)$, the probability density for the particle to diffuse from x' to x during the fictitious time T is the heat-kernel $K(x, x'; T)$. This satisfies the heat equation

$$\partial_T K(x, x'; T) = \Delta_{g^o}^{\text{eff}} K(x, x'; T), \quad (98)$$

where $\Delta_{g^o}^{\text{eff}}$ denotes the usual covariant Laplacian at low energy, which may undergo substantial modifications in the ultra-violet regime. In particular, we are interested in the effective Laplacian at high energy on the flat background ($g_{\mu\nu}^o = \eta_{\mu\nu}$) where the graviton propagates. The heat-kernel is a matrix element of the operator $\exp(T \Delta_g)$ acting on the real Hilbert space of position eigenstates

$$K(x, x'; T) = \langle x' | \exp(T \Delta_{g^o}^{\text{eff}}) | x \rangle. \quad (99)$$

Its trace per volume unit,

$$\begin{aligned} P(T) &\equiv \frac{\int d^D x \sqrt{g^o(x)} K(x, x; T)}{V} \\ &\equiv \frac{\text{Tr} \exp(T \Delta_{g^o}^{\text{eff}})}{V} \end{aligned} \quad (100)$$

can be interpreted as an average return probability. Here, $V \equiv \int d^D x \sqrt{g}$ denotes the total volume. It is acknowledged that $P(T)$ possesses an asymptotic expansion for $T \rightarrow 0$ of the form $P(T) = (4\pi T)^{-D/2} \sum_{n=0}^{\infty} A_n T^n$. The coefficients A_n have a geometric meaning, i.e. A_0 is the volume of the manifold. Knowing $P(T)$, one can recover the dimensionality of the manifold \mathcal{M} as the limit for large T of

$$d_s \equiv -2 \frac{\partial \ln P(T)}{\partial \ln T}. \quad (101)$$

This formula defines the fractal dimension we are going to use.

Omitting the tensorial structure in (34), which does not affect the spectral dimension, we can easily obtain

the heat-kernel. We know that the propagator (in the coordinate space) and the heat-kernel are related by [70–78]

$$\begin{aligned} G(x, x') &= \int_0^{+\infty} dT K(x, x'; T) \\ &= \int \frac{d^D k}{(2\pi)^D} e^{ik(x-x')} \int_0^{+\infty} dT K(k; T), \end{aligned} \quad (102)$$

where

$$G(x, x') = \int \frac{d^D k}{(2\pi)^D} e^{ik(x-x')} \mathcal{O}^{-1}(k)$$

is the Fourier transform of (34). By inverting (102) with respect to the heat-kernel in the momentum space, we get

$$K(k; T) \propto e^{-T k^2 V(k^2/\Lambda^2)^{-1}}. \quad (103)$$

The necessary trace to calculate the average return probability is obtained from the Fourier transform of (103),

$$K(x, x'; T) \propto \int d^D k e^{-T k^2 V(k^2/\Lambda^2)^{-1}} e^{ik(x-x')}. \quad (104)$$

Now we are ready to calculate the average return probability defined in (100) as follows

$$P(T) \propto \int d^D k e^{-T k^2 V(k^2/\Lambda^2)^{-1}}. \quad (105)$$

We then proceed to calculate explicitly the spectral dimension for the three different form factors listed above.

* Form factor 1. At high energy, $V(k^2)^{-1} \sim k^{2\gamma+2N+2}$. Therefore, we can calculate the integral (105) and then the spectral dimension defined in (101) for small T is

$$P(T) \propto T^{-D/(2\gamma+2N+4)} \implies d_s = \frac{D}{\gamma + N + 2}. \quad (106)$$

Since the parameter $\gamma > D_{\text{even}}/2$ or $\gamma > (D_{\text{odd}} - 1)/2$, the spectral dimension is $d_s < 1 \forall D$ and

$$\lim_{D \rightarrow +\infty} d_s = 1 \quad (107)$$

applies, which is a “universal” property of this class of theories. Using the explicit form of the entire function $H(k^2/\Lambda^2)$ given in (40), we calculate the spectral dimension at all energy scales as the fictitious time T varies. Integrating numerically (105), we can plot directly the spectral dimension achieving the graphical result in Fig.6 for $D = 4, 6, 8, 10$ and $\gamma = 3, 4, 5, 6$, respectively.

* Form Factor 2. For simplicity we consider the case $n = 1$, even though for $n > 1$ the result is qualitatively the same. Given the form factor in the momentum space $V(k^2/\Lambda^2) = \exp(-k^2/\Lambda^2)$, the propagator scaling reads

$$\mathcal{O}(k)^{-1} \propto \frac{e^{-k^2/\Lambda^2}}{k^2}, \quad (108)$$

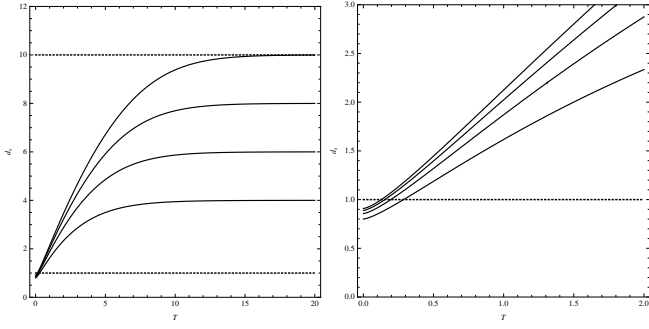


FIG. 6: Plot of the spectral dimension as a function of the fictitious time T for $D = 4, 6, 8, 10$ and the minimal values $\gamma = 3, 4, 5, 6$ in (40). The graph on the right shows that the spectral dimension approximates $d_s = 1$ when increasing D .

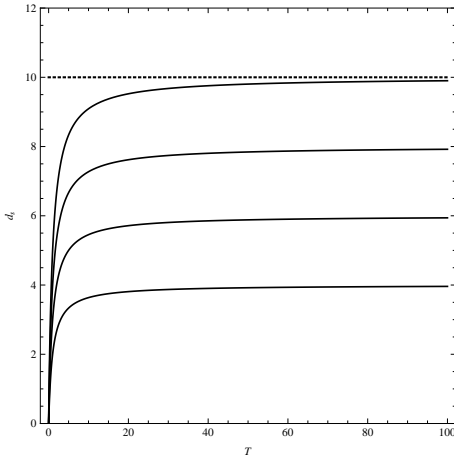


FIG. 7: Plot of the spectral dimension as a function of the fictitious time T for $D = 4, 6, 8, 10$ and form factor $V(z) = \exp(-z)$.

and the heat-kernel can be calculated analytically,

$$K(x, x'; T) = \frac{e^{-\frac{(x-x')^2}{4(T+1/\Lambda^2)}}}{[4\pi(T + \Lambda^{-2})]^{\frac{D}{2}}}, \quad (109)$$

as verifiable by (108). Applying (101), we find that the spectral dimension is

$$d_s = \frac{T}{T + \Lambda^{-2}} D, \quad (110)$$

which clearly goes to zero for $T \rightarrow 0$ and approaches $d_s = D$ for $T \rightarrow +\infty$. A plot of the spectral dimension flow is given in Fig.7 for $D = 4, 6, 8, 10$.

* Form factor 3. In this case the spectral dimension in the ultraviolet regime is $d_s = 2, \forall D$. A plot of the spectral dimension flow is given in Fig.8 for $D = 4, 6, 8, 10$.

It is remarkable to note that for all the three classes of theories parametrized by (γ, n, D) and studied in this section, we always find an accumulation point for the

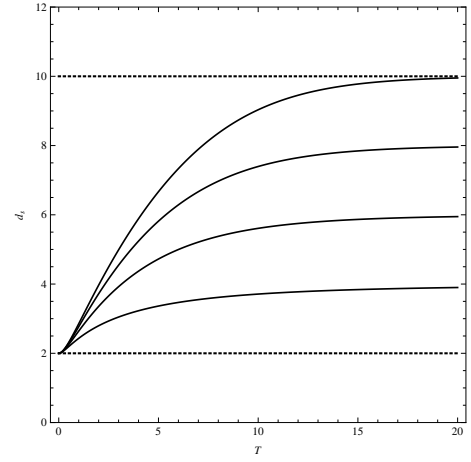


FIG. 8: Plot of the spectral dimension as a function of the fictitious time T for $D = 4, 6, 8, 10$ and form factor $V_3(z)$.

spectral dimension in the ultraviolet regime. In other words, once perturbative renormalizability and unitarity have set the form factors, the spectral dimension in the ultraviolet regime flows to the same “critical point” or “accumulation point” independently from the topological dimension D . From this evidence, we can infer that any consistent theory of quantum gravity must satisfy the following fractal property,

$$d_s \leq 2 \quad \forall D \quad \text{in the ultraviolet regime.} \quad (111)$$

CONCLUDING REMARKS

This study is a synthesis of concepts coming from non-local quantum field theory [27], particle physics, general relativity and string field theory [123–128]. In this article we suggested ways to quantize gravity, relying on the perturbative approach that has been so successful for the other fundamental forces. We introduced a non-local extension of the higher-derivative gravity, which is perturbatively renormalizable and unitary in any dimension D . The four-dimensional theory is easily obtained from the Stelle theory [17] by introducing in the action two entire functions, a.k.a. “form factors”, between the Ricci tensor square and the Ricci scalar square,

$$\begin{aligned} R^2 &\rightarrow R h_0(-\square/\Lambda^2) R, \\ R_{\mu\nu} R^{\mu\nu} &\rightarrow R_{\mu\nu} h_2(-\square/\Lambda^2) R^{\mu\nu}. \end{aligned} \quad (112)$$

In the multidimensional spacetime we preserved the two “delocalization-operators” as in (112) and we implemented a finite number of local operators required (and/or generated) by the quantum consistency of the theory. These local operators $\mathcal{O}_{2n}(\partial g)$ contain $2n$ -derivatives of the metric tensor up to the mass dimension $[\mathcal{O}_{2n}(\partial g)] \leq M^D$. The action may also present other ir-

relevant operators, whose couplings constants have negative mass dimension. The full action reads

$$\begin{aligned}
S = & \int d^D x \sqrt{|g|} \left[2\kappa^{-2} R + \bar{\lambda} \right. \\
& + \sum_{n=0}^N \left(a_n R (-\square_\Lambda)^n R + b_n R_{\mu\nu} (-\square_\Lambda)^n R^{\mu\nu} \right) \\
& + R h_0(-\square_\Lambda) R + R_{\mu\nu} h_2(-\square_\Lambda) R^{\mu\nu} \left. \right] \\
& + \underbrace{O(R^3) \dots + R^{N+2}}_{\text{Finite number of terms}} + \underbrace{O(R^{N+3}) + O(R(-\square_\Lambda)^{N+1} R)}_{\text{Non renormalized operators}}.
\end{aligned} \tag{113}$$

The main reason for introducing the entire functions $h_2(z)$ and $h_0(z)$ is to avoid ghosts (or rather the poltergeists: states of negative norm) and any other new pole in the graviton propagator. The unitarity requirement implies the following entire functions,

$$\begin{aligned}
h_2(z) &= \frac{V(z)^{-1} - 1 - \frac{\kappa^2 \Lambda^2}{2} z \sum_{n=0}^N \tilde{b}_n z^n}{\frac{\kappa^2 \Lambda^2}{2} z} \\
&\quad - O(R(-\square_\Lambda)^{N+1} R), \\
h_0(z) &= - \frac{V(z)^{-1} - 1 + \kappa^2 \Lambda^2 z \sum_{n=0}^N \tilde{a}_n z^n}{\kappa^2 \Lambda^2 z} \\
&\quad - O(R(-\square_\Lambda)^{N+1} R).
\end{aligned} \tag{114}$$

The first set of operators in the last line of (113) is subject to renormalization at quantum level, whereas the second set remains classical as it can be proved by power-counting arguments. Clearly, the non-renormalized operators $O(R(-\square_\Lambda)^{N+1} R)$ in (114) can be eliminated in both the action and the entire functions.

The form factors $V(z)^{-1}$ studied in this paper can essentially show two possible high energy behaviors, either polynomial or exponential. In the first case, the operators $O(R^{N+3})$ may affect the renormalizability of the theory, therefore the polynomial asymptotic degree of $V(z)^{-1}$ has to be increased. In the second case, the same local operators do not thwart the renormalizability of the theory at all.

Let us gather here all the quantities to be measured to define the theory,

$$\begin{aligned}
& (a_n, b_n), \quad 0 \leq n \leq N, \\
& O(c_n R^n), \quad 3 \leq n \leq N+2 \text{ (local relevant operators)}, \\
& V(z), \text{ non-local form factor}, \\
& O(d_n R^n), \quad n \geq N+3 \text{ (local irrelevant operators)}.
\end{aligned} \tag{115}$$

As pointed out throughout the paper, the irrelevant operators can always be introduced in any physical theory as long as they do not invalidate its unitarity and renormalizability. The most important step we need to take is to assess whether the physical measurable quantities are affected or not by such irrelevant operators in (115). If we can assume that no physical quantity is susceptible to such operators, we can then empirically infer that the coupling constants equal “zero”.

The question lingers whether the form factor $V(z)$ is measurable or not. In principle we can treat $V(z)$ as one of the form factors for the scattering of the nucleus by electrons. In a gravitational theory, such measure represents the graviton scattering amplitude, as well as the modifications to the gravitational potential or the light-bending. The four-gravitons amplitude will have the general structure

$$\mathcal{A}_{4g.} = \mathcal{A}_{4g.}(s, t, u; V(s, t, u); \epsilon_{1,2,3,4}), \tag{116}$$

where $\epsilon_{1,2,3,4}$ are the four gravitons polarizations and s, t, u the Mandelstam variables. Since $V(z)$ has to be an entire function, we can falsify the theory by comparing the experimental four-gravitons amplitude with the theoretical prediction (116). A final remark about the tree-level unitarity of the theory has to be put forward at this point. Although we consider this issue still open [2–4, 40], we must also acknowledge that, at high energy, the total cross section of a nonlocal interacted theory must not exceed that of a local one. Intuitively, the reason is that “nonlocal particles” must manifest the transparency property at high energy because of their non-zero size. For example, if the amplitude in the momentum space grows exponentially

$$\mathcal{A} \approx e^{(k^2/\Lambda^2)^\rho}, \quad \rho \geq 1, \tag{117}$$

the total cross-section will satisfy the following upper bound [31],

$$\sigma_{\text{tot}}(s) \leq \text{const. } s^{\rho-1} \log^2(s), \tag{118}$$

which grows logarithmically in the ultraviolet regime if $\rho = 1$.

In this paper, we studied the following three classes of form factors $V(z)$,

$$\text{Form Factor 1. } V_1(z) = e^{-H_\gamma(z)},$$

$$\text{Form Factor 2. } V_2(z) = e^{-z^n}, \quad n \in \mathbb{N}_+,$$

$$\text{Form Factor 3. } V_3(z) = e^{-H_{\gamma=0}(z)},$$

where

$$H_\gamma(z) = \frac{1}{2} [\gamma_E + \Gamma(0, z^{2\gamma+2N+2})] + \log(z^{\gamma+N+1}). \tag{119}$$

We systematically showed the power-counting renormalizability and the tree-level unitarity. The theories defined by the form factors $V_1(z)$ and $V_2(z)$ result to be renormalizable at one loop and finite from two loops upward. More precisely, the theories turn out to be super-renormalizable because the covariant counter-terms have less derivatives than the classical action and the coefficients of the terms with more derivatives do not need any kind of infinity renormalization as synthesized in the first part of this section. However, we argue that a supersymmetric extension of the theory [122] can make it finite at one loop as well. For the third choice $V_3(z)$,

the theory is merely renormalizable, and no other pole beyond the graviton one appears in the propagator.

We solved the linearized equations of motion and we proved that the gravitation potential is regular in $r = 0$ for all the choices of form factors compatible with renormalizability and unitarity. We also included Black hole spherical symmetric solutions omitting higher curvature corrections to the equation of motions. For two out of three form factors ($V_1(z)$ and $V_2(z)$) the solutions are regular and the classical singularity is replaced by a “de Sitter-like core” in $r = 0$. For the third choice $V_3(z)$, black holes are still singular, although the divergence is attenuated.

For $V_1(z)$, we proved that the “Newtonian cosmology” is singularity-free in any dimension D and the Universe spontaneously follows a de Sitter evolution at the “Planck scale” for any matter content (either dust or radiation), since the cosmological constant dominates the effective energy tensor at high energy. In a D -dimensional spacetime the modified Friedmann equation (for $K = 0$ and $\Lambda_{cc} = 0$) reads

$$H^2 = \frac{16\pi G_N}{(D-2)(D-1)} \rho \mathbb{F}_D(a), \quad (120)$$

where $\mathbb{F}_D(a) \approx (a \Lambda)^{2\gamma+2N+2}$ for $a \approx 0$.

Finally, we have provided an extensive analysis of the spectral dimension for any D and for the three classes of theories. In the ultraviolet regime, the spectral dimension takes on different values for the three cases:

$$V_1(z) \implies d_s \lesssim 1,$$

$$V_2(z) \implies n \in \mathbb{N}_+, \quad d_s = 0,$$

$$V_3(z) \implies d_s = 2.$$

where $H_\gamma(z)$ is defined in (119). Once the class of theories compatible with renormalizability and unitarity is defined, the spectral dimension has the same short-distance “critical value” or “accumulation point” for any value of the topological dimension D . This is a “universal” property of the theories here studied.

We would like to conclude this section by identifying some similarities between the second class of super-renormalizable theories and “string field theory”. Using the results found at the end of the Eighties [123–128] and several more recent ideas [49, 129], the string field theory has the following schematic structure for the spacetime bosonic and fermionic fields,

$$S = \int d^D x \left(\frac{1}{2} \phi_i K_{ij}(\square) \phi_j - v_{ijk} \tilde{\phi}_i \tilde{\phi}_j \tilde{\phi}_k \right), \quad (121)$$

where

$$\tilde{\phi}_i \equiv e^{\alpha' \frac{\ln(3\sqrt{3})}{2}} \square \phi_i,$$

$K_{ij}(\square) \approx \square$ for open as well as close bosonic strings, and α' is the inverse mass square in string theory. By a field redefinition [129], the action (121) simplifies to

$$S = \int d^D x \left(\frac{1}{2} \phi_i \square e^{-\alpha' \ln(3\sqrt{3})} \square \phi_j - v_{ijk} \phi_i \phi_j \phi_k \right). \quad (122)$$

We can immediately observe that the kinetic term in (122) has the same scaling of the linearized theory studied in this paper for the exponential form factor $V_2(z) = \exp(-z)$ ($n = 1$). If we expand (113) in powers of the graviton field neglecting the exponential factor in the interaction, the three-graviton vertex is quite similar to the one in (122). However, the general covariance in (113) implies the same leading scaling in the kinetic term as well as in the interaction vertexes and we are unable to get a finite theory at any order in the loop expansion. As already pointed out in this section, one possible loophole to this puzzle could be a supersymmetric extension of the action in (15).

About the finiteness of string theory, we are likely to endorse the following ideas. Due to the presence of the exponential factor, the effective string theory in (122) manifests an asymmetry between the kinetic and the interaction terms. Contrary to our covariant action (113), such asymmetrical state implies that the string theory does not manifest any divergence. The well-known “softness” of the high energy tree-level amplitudes also descends from the same asymmetry.

However, the comparison here proposed can only be qualitative and partial because, unlike the effective string field theory, ours is a general covariant theory. Indeed, general coordinate invariance in string theory can only be achieved through cancellations among contributions from infinitely many interactions terms [126]. However, we do not exclude that a supersymmetric extension of our theory (113) can be framed within “M-theory” as one of its possible vacuums.

APPENDIX: 3D HIGHER-DERIVATIVE QUANTUM GRAVITY

In this section, as a particular toy model, we consider a nonlocal generalization of the 3D higher derivative gravity studied in [109]. The nonlocal action is

$$S = \frac{1}{\kappa^2} \int d^3 x \sqrt{|g|} [\sigma R + R \alpha(\square) R + R_{\mu\nu} \beta(\square) R^{\mu\nu}], \quad (123)$$

where the two “form factors” $\alpha(\square)$ and $\beta(\square)$ are “entire functions” of the covariant D’Alembertian operator. We introduce the following definitions,

$$\alpha(\square) := \alpha_0 + h_0(-\square_\Lambda), \quad \beta(\square) := \beta_0 + h_2(-\square_\Lambda), \quad (124)$$

where $\square_\Lambda := \square/\Lambda^2$ and h_0, h_2 are entire functions. The two form factors have dimensions: $[\alpha(\square)] = [\beta(\square)] = L^2$. The Lagrangian, complete with the gauge fixing and ghost terms, reads

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{GH}}, \quad (125)$$

where \mathcal{L}_g is expressed by (123), and the graviton fluctuation $h^{\mu\nu}$ is defined by

$$\tilde{g}^{\mu\nu} := \sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}. \quad (126)$$

Imposing the BRST invariance on the full Lagrangian (125), we can get the gauge-fixing and ghost terms of the action. The BRST transformation for the fields in (125) appears

$$\begin{aligned} \delta_B g_{\mu\nu} &= -\delta\lambda [g_{\rho\nu}\partial_\mu c^\rho + g_{\rho\mu}\partial_\nu c^\rho + \partial_\rho g_{\mu\nu}c^\rho], \\ \delta_B c^\mu &= -\delta\lambda c^\rho \partial_\rho c^\mu, \\ \delta_B \bar{c}_\mu &= i\delta\lambda \omega(\square) B_\mu, \end{aligned} \quad (127)$$

where c^μ, \bar{c}_ν are the anti-commuting ghosts fields, B_μ is the auxiliary field, $\delta\lambda$ is an anti-commuting constant parameter and $\omega(\square)$ is a weight function of the d'Alembertian operator. The dimensions of the fields are: $[h] = L^{-1/2}$, $[c] = L^y$, $[\bar{c}] = L^{-y-1/2}$, $[B] = L^{-3/2}$, $[\delta\lambda] = L^{1-y}$. The BRST transformation for the graviton field in (126) can be extracted from

$$\begin{aligned} \delta_B \tilde{g}_{\mu\nu} &= \delta\lambda (\tilde{g}^{\mu\rho}\partial_\rho c^\nu + \tilde{g}^{\nu\rho}\partial_\rho c^\mu - \tilde{g}^{\mu\nu}\partial_\rho c^\rho - \partial_\rho \tilde{g}^{\mu\nu}c^\rho \\ &\equiv \delta\lambda \mathcal{D}_\rho^{\mu\nu} c^\rho, \end{aligned} \quad (128)$$

which implies $\delta_B h^{\mu\nu} = \kappa\delta\lambda \mathcal{D}_\rho^{\mu\nu} c^\rho$ [109]. The gauge fixing and ghost Lagrangian can both be expressed as a BRST variation of the following functional

$$\begin{aligned} \mathcal{L}_{GF} + \mathcal{L}_{GH} &= i\delta_B [\bar{c}_\mu (\partial_\nu h^{\mu\nu} - a B^\mu/2)] \frac{1}{\delta\lambda} \\ &= -B_\mu \omega(\square) \partial_\nu h^{\mu\nu} - i\kappa \bar{c}_\mu \mathcal{D}_\rho^{\mu\nu} c^\rho + \frac{a}{2} B_\mu \omega(\square) B^\mu. \end{aligned} \quad (129)$$

To obtain the graviton propagator, we first eliminate the auxiliary field B_μ to get the following gauge fixing Lagrangian,

$$\mathcal{L}_{GF} = -\frac{1}{2a} (\partial_\nu h^{\mu\nu}) \omega(\square) (\partial_\nu h^{\mu\nu}) \quad (130)$$

and then we assemble the quadratic part of (125), namely

$$\begin{aligned} \mathcal{L} &= \frac{1}{4} h^{\mu\nu} \square [P^{(2)}(\square) \square \beta(\square) + \sigma] \\ &+ \square \omega(\square) P^{(1)}/a + P^{(0,s)}((8\alpha(\square) + 3\beta(\square)) - \sigma) \\ &+ 2P^{(0,\omega)}((8\alpha(\square) + 3\beta(\square))\square - \sigma + \omega(\square)/a) + \sqrt{2} \times \\ &(P^{(0,s\omega)} + P^{(0,\omega s)}((8\alpha(\square) + 3\beta(\square))\square - \sigma))_{\mu\nu,\rho\sigma} h^{\rho\sigma}. \end{aligned} \quad (131)$$

In (131) we have introduced the 3D projectors. Using the orthogonality and the completeness property of the projectors, we find the graviton propagator

$$\begin{aligned} D(k) &= \frac{1}{(2\pi)^3 k^2} \left[\frac{P^{(2)}}{\beta(k^2)k^2 - \sigma} + \frac{P^{(0,s)}}{(8\alpha(k^2) + 3\beta(k^2))k^2 + \sigma} \right. \\ &\left. - \frac{a}{\omega(k^2)} \left(P^{(1)} + P^{(0,s)} + \frac{P^{(0,\omega)}}{2} - \frac{\sqrt{2}}{2} (P^{(0,s\omega)} + P^{(0,\omega s)}) \right) \right]. \end{aligned}$$

In the harmonic gauge $\partial_\mu h^{\mu\nu} = 0$ (or $a = 0$), the propagator considerably simplifies to

$$D(k) = \frac{1}{(2\pi)^3 k^2} \left[\frac{P^{(2)}}{\bar{h}_2\left(\frac{k^2}{\Lambda^2}\right)} + \frac{P^{(0,s)}}{\bar{h}_0\left(\frac{k^2}{\Lambda^2}\right)} \right], \quad (132)$$

where the following notation has been introduced,

$$\begin{aligned} \bar{h}_2(z) &:= \beta(z)z\Lambda^2 - \sigma, \\ \bar{h}_0(z) &:= (8\alpha(z) + 3\beta(z))z\Lambda^2 + \sigma, \\ z &:= -\square_\Lambda. \end{aligned} \quad (133)$$

As in the D -dimensional case (31), we choose the entire functions $h_2(z), h_0(z)$ compatibly with the properties (i)-(iii),

$$\begin{aligned} h_2(z) &= -\frac{\sigma(V(z)^{-1} - 1) + \tilde{\beta}_0\Lambda^2 z}{\Lambda^2 z}, \\ h_0(z) &= \frac{4\sigma(V(z)^{-1} - 1) - 8\tilde{\alpha}_0\Lambda^2 z}{8\Lambda^2 z}. \end{aligned} \quad (134)$$

Assuming the theory to be renormalized at a particular scale μ_0 , we identify

$$\tilde{\alpha}_0 = \alpha_0(\mu_0), \quad \tilde{\beta}_0 = \beta_0(\mu_0). \quad (135)$$

In this case $\bar{h}_2 = \bar{h}_0 = V(z)^{-1}$ and the propagator simplifies to

$$D(k) = -\frac{V\left(\frac{k^2}{\Lambda^2}\right)}{(2\pi)^3 \sigma k^2} [P^{(2)} - P^{(0,s)}]. \quad (136)$$

The pole structure of the propagator is the same one as that in the local theory, because $V(z)$ is an entire function with no zeros in the complex plane. In $D = 3$ there are no local degrees of freedom, and therefore the amplitude (46) is identical to zero.

What we have presented in this section is a toy model of modified nonlocal gravity that we think it might be interesting to expand on in connection with other three-dimensional theories studied in recent years [110–121].

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